ADVANCED ALGEBRA

TEACHER'S EDITION

Exploring Regression

G. BURRILL, J. BURRILL, P. HOPFENSPERGER, J. LANDWEHR

DATA-DRIVEN MATHEMATICS

DALE SEYMOUR PUBLICATIONS®

Exploring Least-Squares Linear Regression

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Gail F. Burrill, Jack C. Burrill, Patrick W. Hopfensperger, and James M. Landwehr

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About *Data-Driven Mathematics*

Historically, the purposes of secondary-school mathematics have been to provide students with opportunities to acquire the mathematical knowledge needed for daily life and effective citizenship, to prepare students for the workforce, and to prepare students for postsecondary education. In order to accomplish these purposes today, students must be able to analyze, interpret, and communicate information from data.

Data-Driven Mathematics is a series of modules meant to complement a mathematics curriculum in the process of reform. The modules offer materials that integrate data analysis with secondary mathematics courses. Using these materials helps teachers motivate, develop, and reinforce concepts taught in current texts. The materials incorporate the major concepts from data analysis to provide realistic situations for the development of mathematical knowledge and realistic opportunities for practice. The extensive use of real data provides opportunities for students to engage in meaningful mathematics. The use of real-world examples increases student motivation and provides opportunities to apply the mathematics taught in secondary school.

The project, funded by the National Science Foundation, included writing and field-testing the modules, and holding conferences for teachers to introduce them to the materials and to seek their input on the form and direction of the modules. The modules are the result of a collaboration between statisticians and teachers who have agreed on the statistical concepts most important for students to know and the relationship of these concepts to the secondary mathematics curriculum.

A diagram of the modules and possible relationships to the curriculum is on the back cover of each Teacher's Edition of the modules.

Using This Module

Why the Content Is Important

Studying mathematics involving data brings with it the notion of fitting a line to a data set. The desire to find a *best* line gives rise to a need to understand least-squares regression and correlation. Most calculators and computer software today create the least-squares regression line and with it often display the correlation coefficient. It is because of this widespread availability and the misconceptions that can accompany these topics that this module came to be written.

In this module, students will explore the development of the least-squares regression line and its application. Why it works, when it is appropriate to use it, and how it should be interpreted are at the heart of the module. While investigating the relationship between data and the line and when the least-squares line is the *best* line, students will become aware of the dependence of the least-squares line upon both residuals and a minimum point determined by plotting the sum of the squared residuals against the slope and intercept of that line. They will also learn to appreciate the effect of outliers upon the line. Knowing how to find and interpret the correlation coefficient and understanding the expression *the strength of a linear relationship between two variables* are two of the desired outcomes of this module. Throughout the module, students will find many real-world applications of these two important topics: least-squares regression line and the correlation coefficient.

Content

Mathematics content: Students will be able to:

- Represent linear functions symbolically and graphically.
- Determine and interpret slope and intercepts for linear functions.
- Represent quadratic functions symbolically and graphically.
- Determine the minimum point of a quadratic function.
- Graph the sum of quadratic functions.
- Represent absolute-value functions symbolically and graphically.
- Determine the minimum point of an absolute-value function when possible.
- Graph the sum of absolute-value functions.
- Use summation notation and perform summation arithmetic.
- Use variable notation, including subscripts and superscripts.

Statistics content: Students will be able to:

- Calculate residuals.
- Find the sum of squared residuals.
- Find the absolute mean squared error.
- Work with the correlation coefficients r and r^2.
- Describe the linear relationship between two variables.
- Find least-squares regression lines.

Instructional Model

The instructional emphasis *Exploring Least-Squares Linear Regression,* as in all of the modules in *Data-Driven Mathematics,* is on discourse and student involvement. Each lesson is designed around a problem or mathematical situation and begins with a series of introductory questions or scenarios that can prompt discussion and raise issues about that problem. These questions can involve students in thinking about the problem and help them understand why such a problem might be of interest to someone in the world outside the classroom. The questions can be used in whole-class discussion or in student groups. In some cases, the questions are appropriate to assign as homework to be done with input from families or from others not a part of the school environment.

These opening questions are followed by discussion issues that clarify the initial questions and begin to shape the direction of the lesson. Once the stage has been set for the problem, students begin to investigate the situation mathematically. As students work their way through the investigations, it is important that they have the opportunity to share their thinking with others and to discuss their solutions in small groups and with the entire class. Many of the exercises are designed for groups in which each member does one part of the problem and the results are compiled for final analysis and solution. Multiple solutions and solution strategies are also possible, and it is important for students to recognize these situations and to discuss the reasoning behind different approaches. This will provide each student with a wide variety of ways to build his or her own understanding of the mathematics.

In many cases, students are expected to construct their own understanding by thinking about the problem from several perspectives. They do need, however, validation of their thinking and confirmation that they are on the right track, which is why discourse among students, and between students and teacher, is critical. In addition, an important part of the teacher's role is to help students link the ideas within an investigation and to provide an overview of the "big picture" of the mathematics within the investigation. To facilitate this, a review of the mathematics appears in the summaries.

Each investigation is followed by a Practice and Applications section in which students can revisit ideas presented within the lesson. These exercises may be assigned as homework, given as group work during class, or omitted altogether if students are ready to move ahead.

Pacing/Planning Guide

The table below provides a possible sequence and pacing for the lessons.

LESSON	OBJECTIVES	PACING
Introductory Lesson: Why Draw a Line Through Data?	Discover relationships in a scatter plot by drawing lines through the data points.	$\frac{1}{2}$ class period and homework
Lesson 1: What Is a Residual?	Understand the definition of a residual; find the residuals for a line drawn in a scatter plot.	2 class periods
Lesson 2: Finding a Measure of Fit	Investigate different ways to combine residuals to determine the *best* line using the sum of absolute values of residuals and the sum of the squared residuals.	2 class periods
Lesson 3: Squaring or Absolute Value?	Recognize and describe the graph of quadratic and absolute-value functions; recognize what happens when you combine two or more absolute-value functions or two or more quadratic functions.	1 class period and homework
Lesson 4: Finding the *Best* Slope	Find the slope of a line that minimizes the sum of the squared residuals.	1 class period
Lesson 5: Finding the *Best* Intercept	Investigate the relationship between the intercept and the sum of squared residuals.	1 class period
Lesson 6: The *Best* Slope and Intercept	Investigate how the sum of squared residuals depends jointly on the slope and the y-intercept.	1 class period
Lesson 7: Quadratic Functions and Their Graphs	Find and interpret the x-intercepts of a quadratic equation; find a formula to determine the coordinates of the vertex of a parabola.	1–2 class periods and homework
Lesson 8: The Least-Squares Lines	Understand the mathematics behind the least-squares line.	1–2 class periods and homework
Lesson 9: Using the Least-Squares Regression Line	Find and interpret the least-squares linear-regression line.	1–2 class periods
Lesson 10: Correlation	Find and interpret the correlation coefficient.	3 class periods and homework
Lesson 11: Which Model When?	Recognize the need for different linear models and the impact of outliers on the least-squares regression line.	2–3 class periods and homework
		About 4 weeks total time

Use of Data Sets and Teacher Resources

LESSONS	DATA SETS	RESOURCE MATERIALS
Introductory Lesson: Why Draw a Line Through Data?	Calories and Fat in Hamburgers	
Lesson 1: What Is a Residual?	EPA Fuel Economy BMX Dirt Bikes	*Activity Sheet 1*, Problem 5 *Activity Sheet 2*, Problems 8 and 10
Lesson 2: Finding a Measure of Fit	Films and Box-Office Revenue	*Activity Sheet 3*, Problems 2, 8, and 10 *Lesson 2 Quiz*
Lesson 3: Squaring or Absolute Value?		
Lesson 4: Finding the *Best* Slope		*Activity Sheet 3*, Problem 1
Lesson 5: Finding the *Best* Intercept		*Activity Sheet 3*, Problems 1 and 2 *Activity Sheet 4*, Problems 6 and 7
Lesson 6: The *Best* Slope and Intercept	Slope, Intercept, and Sum of Squared Residuals	*Activity Sheet 3*, Problem 2 *Activity Sheet 5*, Problem 1
Lesson 7: Quadratic Functions and Their Graphs	Algebra-Class Data	*Activity Sheet 6*, Problem 10
Lesson 8: The Least-Squares Line		*Activity Sheet 7*, Problems 3 and 5 *Lesson 8 Quiz*
Lesson 9: Using the Least-Squares Linear-Regression Line	Aircraft-Operating Statistics NFC Passing Leaders Per-Capita Incomes and Prices of Ford Mustangs	*Lesson 9 Quiz*
Lesson 10: Correlation	Breakfast-Cereal Data Toy Prices Chicago Bulls, 1995–1996 Crime Rate and Police Spending	*Activity Sheet 8*, Problems 14 and 15 *Lesson 10 Quiz*
Lesson 11: Which Model When?	Green Bay Packers BMI Anscombe Data NHL 500-Goal Club "Lite" Junk Food Glass and Plastic Waste VCR Prices and Ratings Median Home Prices and Annual Growth	*End-of-Module Test*

Technology

A graphing calculator would be a great advantage for this module, preferably one that creates a least-squares regression line and its corresponding correlation coefficient, the median-fit line. If the calculator has the capacity of creating a spreadsheet, it would be very helpful; if not, one would probably need a computer with spreadsheet capabilities. (A graphing calculator resource section, entitled *Procedures for Using the TI-83 Graphing Calculator,* is included at the end of this module). While it may be possible to accomplish this unit without technology, some of the procedures and the number of graphs required would become tedious and time consuming, and the ideas would get lost in that tedium.

Prerequisites

Students should have experience with drawing lines through a data set and finding the equation as well as drawing a median-fit line and determining its equation. In addition, students should be familiar with the use of List feature of a graphing calculator or the use of a spreadsheet.

If experience with these items is needed, we would suggest *Exploring Linear Relations,* a *Data-Driven Mathematics* module published by Dale Seymour.

INTRODUCTORY LESSON

Why Draw a Line Through Data?

Materials: graph paper, rulers
Technology: graphing calculator
Pacing: $\frac{1}{2}$ class period and homework

Overview

This lesson can be used as an exercise for the entire class with the missing data provided by the teacher, given as a homework assignment to secure the missing data and then as an entire class exercise, or simply as a homework assignment. The focus of this lesson is to discover relationships in a scatter plot by drawing a line through the data and to informally introduce a residual.

Teaching Notes

Depending on the background of the students, this lesson could be done by graphing the scatter plot with paper and pencil and determining the equation of the lines using the point slope form of the equation of a line or by using a graphing calculator. The purpose for drawing lines on scatter plots is to assist in the interpretation and analysis of the data. These lines can help identify important data points, summarize relationships between the variables, and predict the variable on the vertical axis from the variable on the horizontal axis.

Solution Key

Discussion and Practice

1. Answers will vary because students will make their own predictions. A sample answer is given.

Item	Serving Size	Estimated Calories	Actual Calories
Chicken McNuggets	6	200	290
French Fries	Regular size	150	210
Ben & Jerry's Cookie Dough Ice Cream	$\frac{1}{2}$ cup	250	68
Saltine Crackers	5	10	2
Beef Ravioli	1 cup	110	115
Tomato Soup	$\frac{1}{2}$ cup	20	30
Skittles	$1\frac{1}{2}$ oz	40	16
Raisins	$\frac{1}{4}$ cup	200	347
Parmesan Cheese	1 tbsp	20	5
Rice-a-Roni	$2\frac{1}{2}$ oz	110	108
Rice Krispies Cereal	$1\frac{1}{2}$ cup	20	120
Cap'n Crunch Cereal	$\frac{3}{4}$ cup	100	110

STUDENT PAGE 1

INTRODUCTORY LESSON

Why Draw a Line Through Data?

OBJECTIVE

Discover relationships in a scatter plot by drawing lines through the data points.

INVESTIGATE

Estimating Calories

The Food and Drug Administration (FDA) requires nutrition labels on food packages. Below is an example of a label from a box of Lucky Charms breakfast cereal.

Nutrition Facts

Serving Size: 1 cup (30 g)
Servings per Container: about 13

Amount per Serving	Cereal	With $\frac{1}{2}$ cup skim milk
Calories	120	160
Calories from fat	10	15
	% Daily Values	
Total Fat 1 g	2%	2%
Saturated Fat 0 g	0%	0%
Cholesterol 0 mg	0%	1%
Sodium 210 mg	9%	11%
Potassium 55 mg	2%	7%
Total Carbohydrates 25 g	8%	10%
Dietary Fiber 1 g	6%	6%
Sugars 13 g		
Other Carbohydrates 11 g		
Protein 2 g		

Discussion and Practice

Without looking at these labels, how well can you estimate the calories of some selected food items?

1. In the table on page 2 is a list of some food items and their serving sizes. Copy the table. After each item write your estimate for how many calories are in one serving. Use the information above as a guide.

STUDENT PAGE 2

Item	Serving Size	Estimated Calories
Chicken McNuggets	6	_____
French Fries	Regular size	_____
Ben & Jerry's Cookie Dough Ice Cream	$\frac{1}{2}$ cup	_____
Saltine Crackers	5	_____
Beef Ravioli	1 cup	_____
Tomato Soup	$\frac{1}{2}$ cup	_____
Skittles	$1\frac{1}{2}$ oz	_____
Raisins	$\frac{1}{4}$ cup	_____
Parmesan Cheese	1 Tbsp	_____
Rice-a-Roni	$2\frac{1}{2}$ oz	_____
Rice Krispies Cereal	$1\frac{1}{2}$ cup	_____
Cap'n Crunch Cereal	$\frac{3}{4}$ cup	_____

2. How well were you able to estimate the number of calories in one serving of these food items? To help answer this question, use a nutrition book to find the actual number of calories for each item. Then make a scatter plot with your estimate of calories on the horizontal axis and the actual number of calories on the vertical axis.

a. Where will a point lie if your estimate was correct?

b. What line can you draw that represents estimates that are 100% accurate? Write the equation of that line and draw it on the scatter plot.

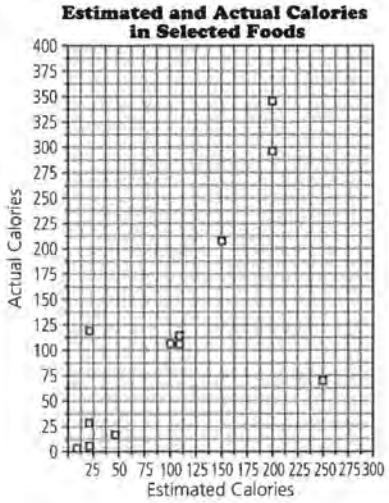
c. How does this line help you decide if you are a good estimator?

d. If the majority of your points were below the line, would you consider your estimates to be overestimates or underestimates? Explain your answer.

e. On your scatter plot, draw a vertical line segment from each point to the line that you have drawn. What do these segments represent?

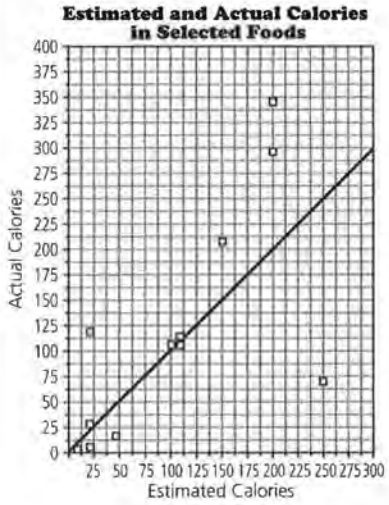
f. Describe how you can find the length of each segment that you drew in part e.

2. Answers will vary. The following sample uses the data given above.



a. Points from correct estimates will lie on the line whose equation is $y = x$.

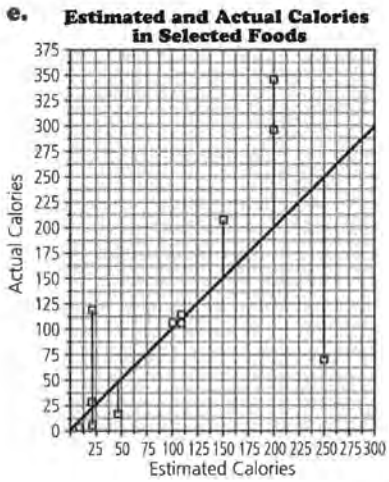
b. $y = x$



c. The better the estimate, the closer to the line the point will be.

d. The points would be over estimates if they are below the line, because the points below the line are points for which $x > y$; that is, the estimate is greater than the actual.

e.

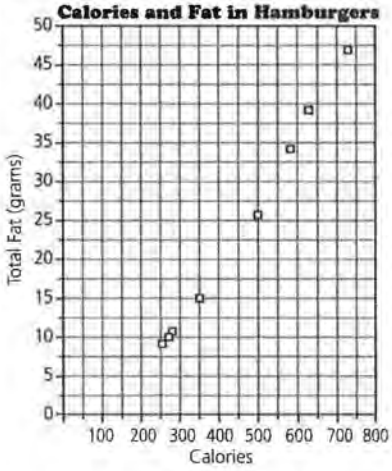


The segments represent the error or difference between the y-value predicted by the equation of the line and the actual y-value given in the data.

f. For any given value of x, subtract the y-value predicted by the equation from the actual y-value given in the data.

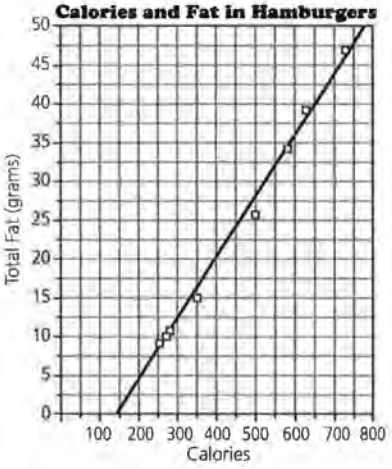
Practice and Applications

3. a.



b. These points indicate a relationship that is almost linear.

4. Answers will vary. Sample:



a. The line enables you to confirm by observation that the relationship is increasing and linear.

b. At $x = 300$ on the graph, draw a vertical line until it intersects the prediction line. At the point of intersection, draw a horizontal line to the y-axis and read the value of the total fat in grams.

STUDENT PAGE 3

Practice and Applications

The table below shows the number of calories and total fat for hamburgers at various fast-food restaurants.

Basic Burgers	Calories	Total Fat (grams)
McDonald's	255	9
Burger King	260	10
Hardee's	260	10
Jack in the Box	267	11
Wendy's Plain Single	350	15
Burgers with the Works		
McDonald's Big Mac	500	26
Jack in the Box Jumbo Jack	584	34
Burger King Whopper	630	39
Hardee's Frisco Burger	730	47

Source: *Consumer Reports*, August, 1994

3. Do you think there is a relationship between calories and total fat content for hamburgers?

- **a.** To investigate any possible relationship in these data, construct a scatter plot with calories on the horizontal axis and total fat on the vertical axis.
- **b.** Is there an association between the variables? Explain.

4. Draw a line on the graph that you think will summarize, or *fit*, the data.

- **a.** How does this line help describe the relationship between the number of calories and total fat content?
- **b.** Describe how you could use the line and predict the fat content in a hamburger if it contained 300 calories.

5. Find an equation of the line that you have drawn by finding two points on the line.

6. What is the slope of the line you drew for Problem 4? Explain the slope in terms of the data.

7. Use your equation from Problem 5 to predict the fat content for a hamburger that has 300 calories.

Summary

We draw lines on scatter plots is to assist in the interpretation and analysis of the data. These lines can help identify important data points, summarize relationships between the variables, and predict the value of the variable on the vertical axis from the value of the variable on the horizontal axis.

5. Answers will vary. The equation of the line drawn in the sample is $y = 0.0777x - 10.785$.
Sample ordered pairs are (295, 12) and (650, 40).

6. Answers will vary. The slope of the line drawn in the sample is 0.0777; it is interpreted as an increase of 100 calories produces as increase of about 7.8 grams of fat in fast-food-restaurant hamburgers.

7. Answers will vary. For the line drawn in the sample, $y = 0.0777(300) - 10.785 = 12.525$, or about 13, grams of fat.

LESSON 1

What Is a Residual?

Materials: graph paper, rulers, *Activity Sheets 1* and *2*
Technology: graphing calculator or computer spreadsheet program
Pacing: 2 class periods

Overview

This lesson introduces the students to the concept of a residual. They are provided with the formal definition and find residuals for a line drawn on a scatter plot. After they are introduced to the concept and determine residuals using paper and pencil, students are encouraged to use a spreadsheet in the determination of the residuals.

Teaching Notes

At first glance, this lesson appears to be quite long. You will notice, however, that a considerable portion of the lesson is dedicated to the use of technology to determine residuals. One option or the other should be used depending on the availability of technology. It is further recommended that it be treated as a whole-class lesson and that a certain amount of teacher modeling and lecturing be used. If the students already have knowledge of residuals, this lesson can be done entirely with technology or used as an assessment of that knowledge.

STUDENT PAGE 4

LESSON 1

What Is a Residual?

How would you like to own a Corvette, or perhaps a Lamborghini?

If the city miles per gallon were known for a certain type of car, could you predict the highway miles per gallon?

The Lamborghini has the lowest city and highway miles per gallon. Does this suggest a general relationship between these two variables?

OBJECTIVES

Understand the definition of a *residual*.

Find the residuals for a line drawn in a scatter plot.

INVESTIGATE

Cars such as the Corvette and Lamborghini are known for their ability to accelerate but not for their high gas mileage. The following table lists eleven sports cars and the 1997 EPA (Environmental Protection Agency) fuel-economy estimate in miles per gallon (mpg) for city and highway driving for each.

Model	City MPG	Highway MPG
Acura NSX	18	24
Alfa Romeo Spider	22	25
Chevrolet Corvette	17	25
Ferrari 355	10	15
Jaguar XJS	17	24
Lamborghini Diablo	9	14
Lotus Esprit V8	15	23
Mazda Miata MX-5	22	28
Nissan 300ZX	18	23
Porsche 911 Carrera	17	25
Toyota MR2	20	27

Source: *1997 Gas Mileage Guide*, United States Department of Energy

Solution Key

Discussion and Practice

1. **a.** If you use either city or highway mileage, the Lamborghini has the worst mileage and the Miata has the best.

b. If you order the cars from best to worst both for the city and the highway mpg, the orders are about the same, which indicates that there may be a relationship of some kind. The difference between the city and highway mpg for each are similar, which may indicate a linear relationship. You might expect that as the city mpg increases, highway mpg does also in about the same ratio.

2. It appears to be linear. For each car, the highway mileage per gallon is greater than the city mileage per gallon. This can be observed graphically, because all of the data points are above the line $C = H$.

3. **a.** About 20; you can sketch in a line that seems to represent the trend in the data and use this line to predict the value.

Reading the graph of the line, students can estimate the value corresponding to 14 for city mpg. Some students may write an equation for the line and use that to find the value. Others may just make a rough estimate looking at the points.

b. There is likely to be some variation, even if students all use a line to approximate the values.

4. If you have access to other data such as cost, you might use that. Based strictly on the information you have already, the only thing you could do is to use the average of the highway mpg for the given cars.

STUDENT PAGE 5

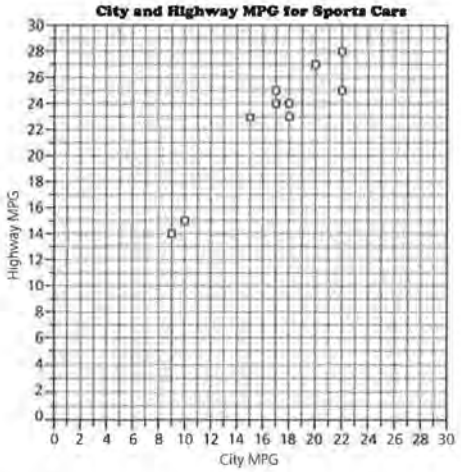
Discussion and Practice

1. Use the table on page 4 to answer the following questions.

a. Which car seems to be the worst in terms of fuel economy? Which has the best gas-mileage rate?

b. What kind of relationship would you expect between the miles per gallon for city driving and for highway driving?

2. Below is a scatter plot of the data. Describe any trends or patterns you observe in the plot. Is the trend consistent with your expectations?



3. There seems to be a relationship between miles per gallon for city driving and miles per gallon for highway driving.

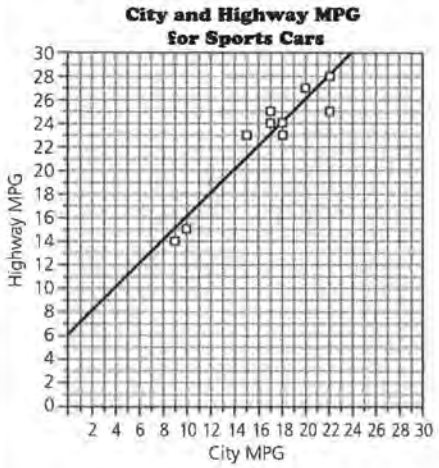
a. A car averages 14 mpg in the city. Use the scatter plot to predict the highway mileage for that car. Explain how you determined your answer.

b. Compare your prediction with other students' predictions. How did the predictions vary?

4. Suppose you want to buy another sports car not on the list but don't know either its highway mileage or its city mileage. Describe how you might predict the highway mileage.

This is an important point and one that will be used later in the lesson on correlation. At this time, be sure students understand that the best they can do numerically is to base their decision on an average.

5. Answers will vary based on student lines; the following set of answers is given using the line $h = 1c + 6$.



a. 28 mpg

b. The line predicts the exact value.

c. Answers depend on class data.

d. Answers will vary. The pattern is quite linear, but using it to predict for miles per gallon less than 9 or more than 24 might be misleading. There is no reason to think the linear pattern will continue; it may increase according to another rule.

6. The error is the difference in observed and predicted y-values, so it is a vertical distance.

STUDENT PAGE 6

Predicting from a Graph

Since there appears to be a linear relationship between city mpg and highway mpg, a line can be drawn on the scatter plot to summarize this relationship. This line can also be used to make predictions.

5. On the scatter plot on *Activity Sheet 1*, draw a line that you think will summarize, or *fit*, the data.

- **a.** The city mileage listed for the Mazda Miata is 22 mpg. How many highway miles per gallon would your line predict for the Miata?
- **b.** The actual number of highway miles per gallon listed for the Mazda Miata was 28. How close was your prediction?
- **c.** Compare your prediction to those made by others in class. Whose was closest?
- **d.** Examine one another's lines. Do you think that the student who was closest also has the line that best summarizes the relationship between city and highway miles per gallon? Justify your answer.

Overall, how well does the line fit the data? The goal is to predict highway miles per gallon when you are given the city miles per gallon. The difference between the actual y-values measured and the predicted y-values determined by the line is called the ***error in prediction***.

6. The error is measured vertically from the point to the line. Why does that make sense?

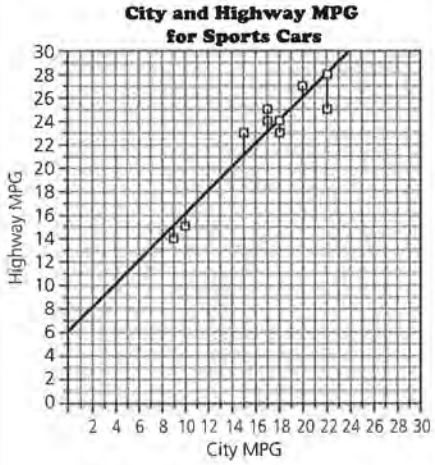
7. **a.** Since both distances are positive, $CD > AB$ means that the line is a better predictor for A than for C.

b. Point C has the greatest residual in absolute value because it has the longest line segment.

c. Points J, H, and K have negative residuals; this means that the predicted values are greater than the actual values for those points.

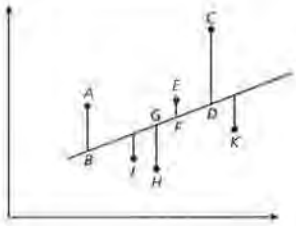
d. They are about the same size, but they are in the opposite direction. This means that for A, the line predicted too low and for H the line predicted too high, but the errors were of equal extent.

8. **a.** For this line, the predictions are too low for the Corvette, Jaguar, Lotus Esprit, Porsche, and Toyota.



b. Each vertical unit on the graph represents 1 mpg.

STUDENT PAGE 7



The scatter plot above uses vertical line segments connecting the data points and the fitted line to show the errors in prediction. These errors are known as ***residuals***. The symbol for the predicted value is $\hat{y}$.

residual = (observed y-value) − (predicted y-value),
symbolically $r = y - \hat{y}$

7. If a data point is above the line, then its y-value, y, is greater than the predicted y-value, $\hat{y}$, and the residual is positive. If the data point is below the line, then its y-value, y, is less than the predicted y-value, $\hat{y}$, and the residual is negative. Use the plot above of points A, E, C, J, H, and K to answer the following questions.

a. $CD > AB$. What does that tell you?

b. Which data point has the greatest residual, in absolute value? How can you tell?

c. List the data points that have negative residuals. What does a negative residual tell you about your prediction?

d. Comment on this sentence: *The residual for A is the same as the residual for H.*

8. Return to the plot of the city and highway miles per gallon.

a. Draw the vertical segments that represent the residuals for your line. For which cars are your predictions too low?

b. What does each vertical unit on the graph represent?

(8) c. See table below.

d. The residual for (15, 23) from the line in part a is +2, which means that the prediction was 2 mpg too low.

e. If all the residuals are small, the prediction line is very accurate.

STUDENT PAGE 8

c. Use the graph of your fitted line to find the predicted highway miles per gallon for each city mile per gallon. Then find the residual for each data point. Record your results in a table like that below or in the first table on *Activity Sheet* 2.

(City MPG, Hwy. MPG)	Predicted Hwy. MPG	Residual
(18, 24)	_____	_____
(22, 25)	_____	_____
(17, 25)	_____	_____
(10, 15)	_____	_____
(17, 24)	_____	_____
(9, 14)	_____	_____
(15, 23)	_____	_____
(22, 28)	_____	_____
(18, 23)	_____	_____
(17, 25)	_____	_____
(20, 27)	_____	_____

d. What is the residual for (15, 23)? What does it represent?

e. If all residuals are small, how accurate is the prediction using the line for these cars?

Summary

A residual for a given x-value is the difference between the observed y-value, y, and the predicted y-value, $\hat{y}$, for that x. The observed y is the y-value for the given x. A residual has the same unit as the y-values. Each residual calculated above was in miles per gallon. Residuals also have a direction, either positive or negative, depending upon their relationship to the line drawn through the data.

Predicting from an Equation

Instead of using the graph of the fitted line to find residuals, you can use an equation of that line.

(City MPG, Hwy. MPG)	Predicted Hwy. MPG	Residual
(18, 24)	24	0
(22, 25)	28	–3
(17, 25)	23	2
(10, 15)	16	–1
(17, 24)	23	1
(9, 14)	15	–1
(15, 23)	21	2
(22, 28)	28	0
(18, 23)	24	–1
(17, 25)	23	2
(20, 27)	26	1

9. **a.** Answers will vary; sample: $y = x + 6$.

b. For this sample: 24 mpg

c. The residual is –1. The prediction was too high.

10. **a.** Answers will vary. Using the same line as before, solutions will be the same as those for Problem 8.

b. 2

STUDENT PAGE 9

9. Pick two ordered pairs on the fitted line that you drew on the scatter plot of city mpg and highway mpg.

a. Use your ordered pairs to write an equation of the line.

b. Use your equation to predict the highway mileage for a Nissan 300ZX rated by the EPA for city mileage at 18 mpg.

c. The listed highway mileage for the Nissan is 23 mpg. Find the residual. Did your equation predict too low or too high a highway mileage?

10. Use your equation from Problem 9a to find the predicted highway miles per gallon for each given city miles per gallon and the corresponding residual for that data point.

a. Record your results in a table like that below or in the second table on *Activity Sheet* 2.

(City MPG, Hwy. MPG)	Predicted Hwy. MPG	Residual
(18, 24)	_____	_____
(22, 25)	_____	_____
(17, 25)	_____	_____
(10, 15)	_____	_____
(17, 24)	_____	_____
(9, 14)	_____	_____
(15, 23)	_____	_____
(22, 28)	_____	_____
(18, 23)	_____	_____
(17, 25)	_____	_____
(20, 27)	_____	_____

b. What is the residual for a city mileage of 15 mpg?

Summary

A residual can be found both graphically and algebraically. Graphically, a residual is the length of the vertical segment between a data point and the fitted line. Algebraically, a residual is the difference for a given x between the observed y-value, y, and the predicted y-value, $\hat{y}$, using the equation of the fitted line:

$$\text{residual} = \text{observed } y - \text{predicted } y$$
$$r = y - \hat{y}$$

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Using Technology to Find Residuals

Calculating residuals can be a long process. A spreadsheet or a graphing calculator allows you to easily calculate, compare, and work with residuals. Throughout this unit, Option A shows you how to use a spreadsheet, Option B a graphing calculator. Suppose the line you have drawn, in slope-intercept form, is $y = 1.2x + 3.5$.

Option A: Spreadsheet

The following shows how a spreadsheet can be set up to find the residuals. Enter the slope of 1.2 in Cell B1 and the intercept 3.5 in Cell B2. After you have typed the equation of the fitted line in C4 and the rule for the residuals in D4, use the fill down command in both columns to calculate the values. The formula in C4 uses the cell locations B1, A4, and B2.

	A	B	C	D
1	Slope =	1.2		
2	Intercept =	3.5		
3	City MPG	Hwy. MPG	Predicted Hwy. MPG	Actual – Predicted
4	18	24	=$B$1*A4+$B$2	=B4–C4
5	22	25		
6	17	25		
7	10	15		
8	17	24		
9	9	14		
10	15	23		
11	22	28		
12	18	23		
13	17	25		
14	20	27		

11. Create a spreadsheet using the format above.

a. Why are some of the values in column D of your spreadsheet negative? What do these values represent?

b. Change the value of the slope in the spreadsheet. What effect did this change have on the individual residuals?

11. See below for sample spreadsheet.

a. The values in column D are the residuals. The negative values mean that for those cars, the line used makes a prediction of a highway value that is greater than the actual value.

b. Answers will vary. Most likely some residuals will grow and others will shrink. Students may want to experiment to get values producing many –1s, zeros, and 1s.

	A	B	C	D
1	**Slope =**	**1.2**		
2	**Intercept =**	**3.5**		
3	**City MPG**	**Hwy. MPG**	**Predicted Hwy. MPG**	**Actual – Predicted**
4	18	24	25.1	–1.1
5	22	25	29.9	–4.9
6	17	25	23.9	1.1
7	10	15	15.5	–0.5
8	17	24	23.9	0.1
9	9	14	14.3	–0.3
10	15	23	21.5	1.5
11	22	28	29.9	–1.9
12	18	23	25.1	–2.1
13	17	25	23.9	1.1
14	20	27	27.5	–0.5

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12. a. The predicted values defined by that equation for all values in L1

b. The predicted value of the equation when x is replaced by 17

c. The residuals, that is, the differences between the predicted and actual values

Option B: Calculator

Another method to find residuals is to use a graphing calculator. The steps below describe how to calculate the residuals on a TI-83. Enter the equation of the line $y = 1.2x + 3.5$ into Y1 = . Then select **STAT EDIT**.

Type the city miles per gallon in List 1, L1.

Type the highway miles per gallon in List 2, L2.

Define L3 = Y1(L1) by moving the cursor to the top of L3.

Type the following: " Y1(L1) ".

Quotation marks are part of the typing.

L1	L2	L3
18	24	
22	25	
17	25	
10	15	
17	24	
9	14	
15	23	
" Y1 (L1) "		

(Note: Only the first seven entries appear in L1 and L2; remember there are eleven entries in each List.)

Place the cursor on L4. Define L4 by typing " L2 – L3 " with the cursor above L4 as pictured below.

L2	L3	L4
24	25.1	
25	29.9	
25	23.9	
15	15.5	
24	23.9	
14	14.3	
15	21.5	
L4 = " L2–L3 "		

12. Answer the following questions.

a. What does the formula you used to define L3 calculate?

b. What does the entry in L3(3) represent?

c. What will the entries in L4 represent?

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13. Enter the data and follow the steps above to find L3 and L4.

a. Why are some of the values in column L4 negative?

b. Change the equation that you have entered in Y1 = . Did the residuals increase or decrease?

14. Write a paragraph summarizing what a residual represents and how to find the residuals for a data set and a line.

Practice and Applications

15. In BMX dirt-bike racing, jumping, or *getting air*, depends on many factors: the rider's skill, the angle of the jump, and the weight of the bike. Here are data about the maximum heights for various bike weights for the same rider.

Weight (pounds)	Height (inches)
19.0	10.35
19.5	10.30
20.0	10.25
20.5	10.20
21.0	10.10
22.0	9.85
22.5	9.80
23.0	9.79
23.5	9.70
24.0	9.60

Source: Statistics Across the Curriculum

a. Make a scatter plot of (weight, height). Draw a line on the graph that you think will fit the data.

b. Predict the height a 21.5-pound bike could clear. Explain how you made your prediction.

c. Predict the height a 35-pound bike could clear. Do you think there is a 35-pound bike?

d. Use either a spreadsheet or graphing calculator to find the residuals and explain what they represent.

16. Compare your line and residuals with those of another student.

a. What conclusions can you make about the residuals?

b. What did you do to come up with your conclusion? What evidence can you give to support your conclusion?

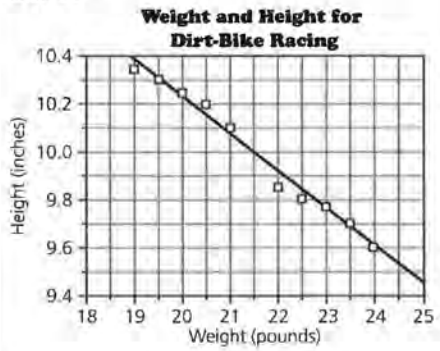
13. a. Those are the points with negative residuals, which means that the predicted value was larger than the observed value.

b. Answers will vary. Most likely some residuals will grow and others will shrink. Students may want to experiment to get values producing many –1s, zeros, and 1s.

14. Sample: The residual is the difference between the observed value and the value predicted by a line that seems to fit the data. The residuals can be found graphically, by drawing the line and then looking at the vertical distance from each point to the line. If the point is above the line, the residual is positive; if the point is below the line, the residual is negative. The residual can also be found by making a table with the observed and predicted values and then subtracting. In this case, the predicted value can be found either by estimating directly from the graph or by writing an equation for the line and using that equation. Finally, a spreadsheet program or graphing calculator can be used. In this case, an equation for the line is needed and columns can be set up to find the predicted value and the residual.

Practice and Applications

15. a.



b. Using the line in part a, a 21.5-lb bike could jump 10 inches. This could be obtained by projecting vertically from the value of $x = 21.5$ on the x-axis and reading the y-value by projecting the intersection horizontally to the y-axis.

c. About 8 inches; the graph has to be extended to determine this value. Most likely there is no 35-lb bike.

d. For the line in part a, the residuals are as follows:

Weight	Residual	Weight	Residual
19.0	–0.05	22.0	–0.08
19.5	0	22.5	–0.05
20.0	0.015	23.0	0
20.5	0.05	23.5	0
21.0	0.03	24.0	–0.01

These represent the differences between the observed and the predicted values.

16. Answers will vary. Results depend on class data.

LESSON 2

Finding a Measure of Fit

Materials: graph paper, rulers, *Activity Sheet 3, Lesson 2 Quiz*
Technology: graphing calculator or computer spreadsheet program
Pacing: 2 class periods

Overview

This lesson investigates how residuals can be used to determine how well a line summarizes a relationship between two variables. In Lesson 1, students found that one way to determine how accurately a line predicted results was to look at the residuals. If the residuals were small, students may have felt that their line summarized the data very well. But the residuals change for each new line drawn. Because it is not possible to decide whether the line fits overall on the basis of one or two data points and their residuals, it is reasonable to look at some methods that combine residuals of a given fitted line into a single measure. A single measure can then be helpful in comparing different lines. Two such single measures are the sum of the absolute values of the residuals and the sum of the squares of the residuals. These two measures and their determination are investigated in this lesson.

Teaching Notes

The ideas and concepts covered in this lesson are probably the most important in the entire unit. The relationship between the line and these two measures is a necessary understanding. Be careful to ensure that the students understand the basic ideas before moving ahead.

Part of this lesson is devoted to the use of technology to determine the sum of the absolute values of the residuals and the sum of the squares of the residuals. It makes the lesson appear longer than it really is since you need not cover both methods.

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LESSON 2

Finding a Measure of Fit

Did you ever notice that around the time that votes are cast for the Academy Awards, Hollywood releases a number of new films?

Many times the major film companies will show these films in a great number of theaters throughout the country. Why do you think they do this?

Do you think there is a relationship between the number of screens on which a movie is shown and the amount of money taken in at the box office?

INVESTIGATE

Movies often have unusual titles. Did you ever hear of *Fried Green Tomatoes?* What do you think *Stop or My Mom Will Shoot* was all about?

The table on page 14 contains the information on the top ten films for the weekend of February 28 to March 1, 1992, about one month before the presentation of the Academy Awards. The box-office revenue column is the amount of money that the movie *grossed*, or took in, in units of $10,000.

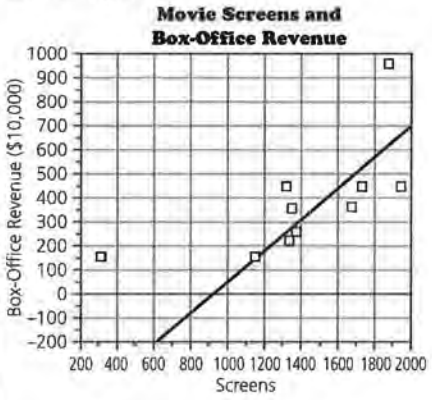
OBJECTIVE

Investigate different ways to combine residuals to determine the *best* line using the sum of the absolute values of the residuals and the sum of the squared residuals.

Solution Key

1. **a.** $3,530,000
   b. $1575.04 per screen
   c. About 353,333 people

2. Sample:



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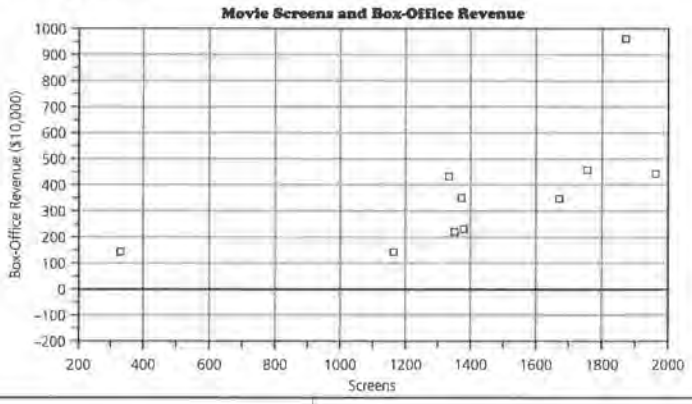
Film	Number of Screens	Box-Office Revenue ($10,000s)
Wayne's World	1878	964
Memoirs of an Invisible Man	1753	460
Stop or My Mom Will Shoot	1963	448
Fried Green Tomatoes	1329	436
Medicine Man	1363	353
The Hand That Rocks the Cradle	1679	352
Final Analysis	1383	230
Beauty and the Beast	1346	212
Mississippi Burning	325	150
The Prince of Tides	1163	146

Source: Entertainment Data Inc. and *Variety*, 1992

1. Use the data in the table to answer the following questions.
   a. How much money did *Medicine Man* gross during that weekend?
   b. On the average, how much money did *Beauty and the Beast* gross per screen?
   c. In 1992, the average ticket price for a movie was about $6. Use that average price for a ticket to estimate the number of people who saw *Beauty and the Beast* over the 3-day period.

2. On the first plot on *Activity Sheet 3*, which contains the scatter plot below, draw a line that you think could be used to predict box-office revenue from the number of screens showing a movie.



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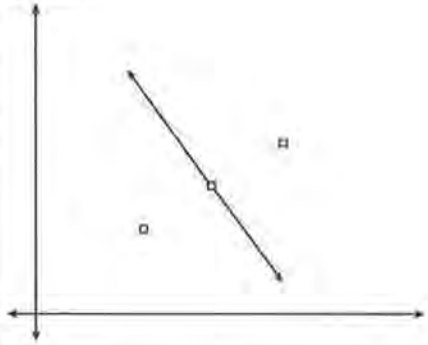
(2) a. and b. Answers will vary. Students may use the residuals in some way, such as adding the residuals, adding the absolute value of the residuals, comparing the number of positive and negative residuals, or comparing the difference between the residuals for each point.

3. a.



b. The sum of the residuals is about zero.

c. Even if the sum of the residuals is zero, it may not be a good line. Because some residuals are positive and some are negative, they may cancel each other and result in a sum of zero when the residuals are quite large. Note: If students think that a sum of zero would mean a good fit, part b should help clear up the misconception.

a. Compare your line to that of a classmate. Which line do you think is better?

b. What criteria did you use to make the decision?

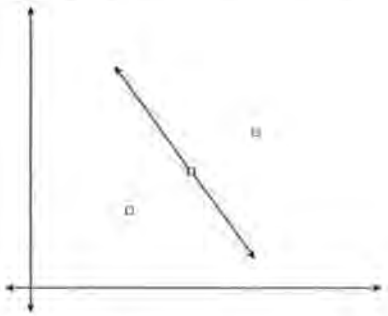
The investigations in this lesson will show you how residuals can be used to determine how well a line summarizes a relationship between two variables. In Lesson 1, you found that looking at residuals is one way to determine how accurately your line predicted outcomes. If the residuals were small, you may have felt that your line summarized the data very well. But the residuals change for each new line you draw. Because it is not possible to decide whether the line fits overall on the basis of one or two data points and their residuals, it is reasonable to look at some methods that combine residuals of a given fitted line into a single measure. A single measure can then be helpful in comparing different lines.

Sum of Residuals

3. One method of combining the residuals is to find the sum.

a. Draw in the residuals for the following plot and line.

b. Estimate the sum of the residuals for the following plot and line.



c. Comment on this statement: *If the sum of the residuals is zero, the line is a good fit for the data.*

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The sign of the residual indicates whether a data point is above or below the fitted line but seems to cause a problem with the sum of the residuals. There are two simple mathematical operations that eliminate negative signs: take the absolute value or take the square.

Consider the data for the box-office revenue and the movie screens. Suppose you drew a line and found the residuals. You could consider the sum of the absolute value of the residuals, written as $\Sigma|\text{residuals}|$, or you could consider the sum of the squares of the residuals, written $\Sigma(\text{residuals})^2$.

4. Suppose the $\Sigma|\text{residuals}| = 3{,}900$ for the box-office data, and $\Sigma(\text{residuals})^2$ is 25,000.

a. What is the average of the absolute values of the residual? This value is called the ***average absolute residual.*** What does the average absolute residual tell you about predicting the revenue given the number of screens?

b. What is the average of the squares of the residuals? This value is called ***average squared residual.***

c. The square root of the average squared residual is called the ***root mean squared error.*** When predicting the revenue given the number of screens, the root mean squared error gives an indication of the amount of error in the predictions. Find the root mean squared error for the residuals in part b.

What are the sums of the absolute values and squared residuals for your line? How do they compare to those from the lines drawn by others in class? As in Lesson 1, either work with a spreadsheet similar to the one shown on page 17 or use a graphing calculator with a list function.

4. a. The average absolute residual is \$390. This means that on the average a prediction would be off by \$390.

b. The average squared residual is \$2500. This means that on the average the square of the difference between the actual value and the predicted value would be \$2500.

c. The root mean squared error for the residuals above is \$50.

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Option A: Spreadsheet

The spreadsheet below is set up to give the results when the line $y = 0.4x - 93$ is used to predict the box-office revenue, y, from the number of screens, x.

	A	B	C	D	E	F
1	**Slope =**	**0.4**				
2	**Intercept =**	**–93**				
3	**Screens**	**Box-Office Revenue**	**Predicted Revenue**	**Residual**	**Absolute Value of Residual**	**Square of Residual**
4	1878	964	=$B$1*A4+$B$2	=B4–C4	=abs(D4)	=(D4)^2
5	1753	460				
6	1963	448				
7	1329	436				
8	1363	353				
9	1679	352				
10	1383	230				
11	1346	212				
12	325	150				
13	1163	146				
14	**Sum of absolute residuals =**				=Sum (E4:E13)	
15	**Sum of squared residuals =**					=Sum (F4:F13)

The results of this spreadsheet are shown on page 18.

5. a. The spreadsheet multiplied A4(1878) by B1(0.4) and then added B2(–93) to get 658.2. This is the predicted value for 1878.

b. The spreadsheet took the absolute value of D4, in this case, the same value. It represents the distance from the line (predicted value) to the point (actual value).

c. This is the square of the value in E4. This is the square of the distance from the line (predicted value) to the point (actual value).

d. This is the sum of the absolute values of the residuals; it is the sum of E4 through E13.

e. The line seems to predict values that are too large.

f. The average absolute residual is 182.9 and the root mean squared error is 201.9. These values represent, on the average, the error expected in the prediction using the given slope and intercept.

6. An equation for the line drawn in this sample is $y = 0.64x - 580$.

a. For the line above, $\Sigma|\text{residuals}| = 1705.88$ and $\Sigma(\text{residuals})^2 = 510519.995$.

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	A	B	C	D	E	F
1	**Slope =**	**0.4**				
2	**Intercept =**	**–93**				
3	**Screens**	**Box-Office Revenue**	**Predicted Revenue**	**Residual**	**Absolute Value of Residual**	**Square of Residuals**
4	1878	964	658.2	305.8	305.8	93513.64
5	1753	460	608.2	–148.2	148.2	21963.24
6	1963	448	692.2	–244.2	244.2	59633.64
7	1329	436	438.6	–2.6	2.6	6.76
8	1363	353	452.2	–99.2	99.2	9840.64
9	1679	352	578.6	–226.6	226.6	51347.56
10	1383	230	460.2	–230.2	230.2	52992.04
11	1346	212	445.4	–233.4	233.4	54475.56
12	325	150	37	113	113	12769
13	1163	146	372.2	–226.2	226.2	51166.44
14	**Sum of absolute residuals =**				1829.4	
15	**Sum of squared residuals =**					407708.52

5. Refer to the results in the spreadsheet to answer the following.

a. Explain how the spreadsheet calculated the value of 658.2 in cell C4. What does this value represent?

b. Explain how the spreadsheet calculated the value of 305.8 in cell E4. What does this value represent?

c. Explain how the spreadsheet calculated the value of 93513.64 in cell F4. What does this value represent?

d. What does the value found in cell E14 represent?

e. The residuals are almost all negatives. What does this tell you about the line?

f. Find the value of the average absolute residual and the root mean squared error. What do these values represent?

6. Find an equation of the line you drew on the movie screens and revenue plot.

a. Change the slope and the intercept on the spreadsheet to match your line's slope and intercept, then record the values $\Sigma|\text{residuals}|$ and $\Sigma(\text{residuals})^2$ for your line.

(6) b. For the equation above, the root mean squared error is 225.95. This means that one would expect to be off by about 225.95 times \$10,000 per movie when using the line to predict.

c. Answers will vary.

7. a. The calculator substituted 1878 into Y1 to get 658.2.

b. The calculator subtracted L3(1) from L2(1).

c. The calculator squared the value of L4(1).

d. The line seems to predict values that are too large.

e. $\Sigma|\text{residuals}| = 1829.4$; $\Sigma(\text{residuals})^2 = 407{,}708.52$

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b. What is the root mean squared error for your line? What does this tell you about the typical error in predicting the revenue from the number of screens?

c. How does your line seem to summarize the relationship between the number of movie screens and the box office revenue compared to the lines drawn by others in class?

Option B: Calculator

Enter the equation in Y1: $Y1 = 0.4x - 93$.

Define: L3 as " Y1(L1) ",

L4 as " L2 – L3 ",

L5 as " abs (L4) ", and

L6 as " L4^2 ".

Screens	Box Office	Predicted	Residual	Absolute	Square
L1	L2	L3	L4	L5	L6
1878	964	658.2	305.8	305.8	93514
1753	460	608.2	−148.2	148.2	21963
1963	448	692.2	−244.2	244.2	59634
1329	436	438.6	−2.6	2.6	6.76
1363	353	452.2	−99.2	99.2	9840.6
1679	352	578.6	−226.6	226.6	51348
1383	230	460.2	−230.2	230.2	52992
L3 = " Y1(L1) "					

7. Use the above results to answer the following.

a. Explain how the calculator found the value of 658.2 in L3(1).

b. Explain how the calculator found the value of 305.8 in L4(1).

c. Explain how the calculator found the value of 93,514 in L6(1).

d. The residuals are almost all negatives. What does this tell you about the line?

e. To find the sum of the absolute or squared residuals, use STAT/CALC/1-Var Stats and select the appropriate list. What are $\Sigma|\text{residuals}|$ and $\Sigma(\text{residuals})^2$?

8. One possible line is $y = 0.64x - 580$.

a. For the line above, $\Sigma|\text{residuals}| = 1705.88$; $\Sigma(\text{residuals})^2 = 510519.995$.

b. For the equation above, the root mean squared error is 225.95. This means that one would expect to be off by about 225.95 (times \$10,000) when using the line to predict.

c. Answers will vary.

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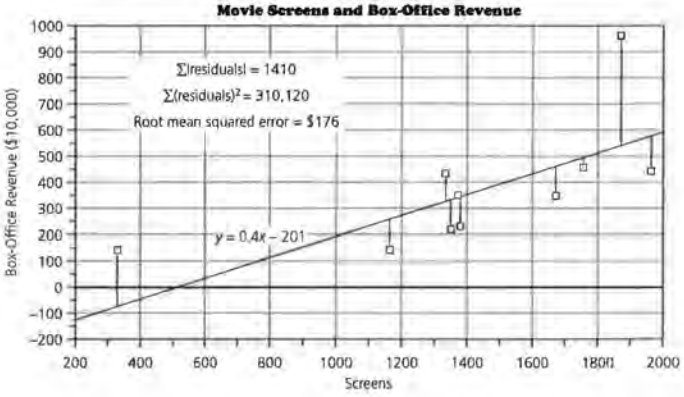
8. Find an equation of the line you drew on the movie-screens-and-revenue plot.

a. Change the equation in Y1 to your equation. Then record the values $\Sigma|\text{residuals}|$ and $\Sigma(\text{residuals})^2$ for your line.

b. Find the value of the root mean squared error. What does this tell you about the typical error in predicting the revenue from the number of screens?

c. How does your line seem to summarize the relationship between the number of movie screens and the box-office revenue compared to the lines drawn by others in class?

Study the plot below. Remember that a residual is the difference between the observed y-value and the predicted y-value for a given value of x. It can be represented by the vertical distance between the line and the data point. The sum of the absolute value of the residuals or the sum of the squares of the residuals will be smallest in the line that fits a data set best.



9. Answers will vary.

10. a., b., and c. Answers will vary.

11. One method is to minimize the sum of the absolute residuals, and the other is to minimize the sum of the squared residuals.

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So far, the slope and intercept for a line, the sum of the absolute values of the residuals, and the sum of the squared residuals for that line have been determined. To find the *best* line, find a line that will minimize the sum of the absolute residuals and a line that will minimize the sum of the squared residuals. Which line is better? Will the same line minimize both sums?

9. Was $\Sigma|\text{residuals}|$ for your line less than the value listed in the figure above? Was the $\Sigma(\text{residuals})^2$ smaller? Do you think your line is a *better* line than the one shown above? Explain.

10. From the lines of other groups, collect the slope, y-intercept, $\Sigma|\text{residuals}|$ and $\Sigma(\text{residuals})^2$.

- **a.** If the definition of the *best* line is the one that minimizes the $\Sigma|\text{residuals}|$, what is the equation of the line from your group that has the least $\Sigma|\text{residuals}|$? Graph this line on the second plot on *Activity Sheet 3*.
- **b.** If the definition of the best line is the one that minimizes the $\Sigma(\text{residuals})^2$, what is the equation of the line that has the least $\Sigma(\text{residuals})^2$? Graph this line on the second plot on *Activity Sheet 3*.
- **c.** Are the two lines identical? If not, explain why there are two lines that could be the *best* line to summarize data presented on a scatter plot.

11. Describe the two methods that can be used to decide which line better summarizes data presented on a scatter plot.

Summary

When finding an equation of the *best* line, it is possible to arrive at different answers depending on the definition that is used. In this section, two definitions were used to define *best* line. One definition defined *best* as the line that minimized the sum of the absolute values of the residuals. Another definition minimized the sum of the squares of the residuals. Lesson 3 will compare these two definitions and discuss which definition is better.

Practice and Applications

12. a. Answers will vary; samples are given.

	Slope	*y*-intercept
Line 1:	–0.15	13.2
Line 2:	–0.18	13.85
Line 3:	–0.14	13

b. Answers will vary; samples are given on the following page.

c. The equation from these three that minimizes the sum of the absolute values is $y = -0.15x + 13.2$. The average absolute residual is 0.034.

d. The equation that minimizes the sum of squared residuals is $y = -0.15x + 13.2$. The root mean squared error is 0.0407.

e. $-0.15(20.5) + 13.2 = 10.125$ inches

f. Answers will vary.

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Practice and Applications

12. Recall that in BMX dirt-bike racing, jumping depends on many factors: the rider's skill, the angle of the jump, and the weight of the bike. Here again are the data presented in Lesson 1 about the maximum height for various bike weights.

Weight (pounds)	Height (Inches)
19.0	10.35
19.5	10.30
20.0	10.25
20.5	10.20
21.0	10.10
22.0	9.85
22.5	9.80
23.0	9.79
23.5	9.70
24.0	9.60

Source: *Statistics Across the Curriculum*

a. Find the slopes and the intercepts of three different lines that you think might summarize the data. Then find an equation of each line.

b. Create a spreadsheet similar to the one shown in this lesson or use a graphing calculator to find the sum of the absolute values of the residuals and the sum of squared residuals for each line.

c. Which equation minimizes the sum of the absolute values? Find the average absolute residual.

d. Which equation of the line minimizes the sum of squared residuals? Find the root mean squared error.

e. Use the equations from parts c and d to predict the height for a 20.5-pound bike.

f. Which of the two equations do you think is better at predicting maximum height for a bike? Explain your answer.

Slope =	**–0.15**				
Intercept =	**13.2**				
Weight (lb)	**Height (in.)**	**Predicted Height**	**Residual**	**Absolute**	**Square**
19	10.35	10.35	0	0	0
19.5	10.3	10.275	0.025	0.025	0.000625
20	10.25	10.2	0.05	0.05	0.0025
20.5	10.2	10.125	0.075	0.075	0.005625
21	10.1	10.05	0.05	0.05	0.0025
22	9.85	9.9	–0.05	0.05	0.0025
22.5	9.8	9.825	–0.025	0.025	0.000625
23	9.79	9.75	0.04	0.04	0.0016
23.5	9.7	9.675	0.025	0.025	0.000625
24	9.6	9.6	0	0	0
Sum of absolute residuals				0.34	
Sum of squared residuals					0.0166

Slope =	**–0.18**				
Intercept =	**13.85**				
Weight (lb)	**Height (in.)**	**Predicted Height**	**Residual**	**Absolute**	**Square**
19	10.35	10.43	-0.08	0.08	0.0064
19.5	10.3	10.34	-0.04	0.04	0.0016
20	10.25	10.25	0	0	0
20.5	10.2	10.16	0.04	0.04	0.0016
21	10.1	10.07	0.03	0.03	0.0009
22	9.85	9.89	-0.04	0.04	0.0016
22.5	9.8	9.8	0	0	0
23	9.79	9.71	0.08	0.08	0.0064
23.5	9.7	9.62	0.08	0.08	0.0064
24	9.6	9.53	0.07	0.07	0.0049
Sum of absolute residuals				0.46	
Sum of squared residuals					0.0298

Slope =	**–0.14**				
Intercept =	**13**				
Weight (lb)	**Height (in.)**	**Predicted Height**	**Residual**	**Absolute**	**Square**
19	10.35	10.34	0.01	0.01	0.0001
19.5	10.3	10.27	0.03	0.03	0.0009
20	10.25	10.2	0.05	0.05	0.0025
20.5	10.2	10.13	0.07	0.07	0.0049
21	10.1	10.06	0.04	0.04	0.0016
22	9.85	9.92	-0.07	0.07	0.0049
22.5	9.8	9.85	-0.05	0.05	0.0025
23	9.79	9.78	0.01	0.01	0.0001
23.5	9.7	9.71	-0.01	0.01	0.0001
24	9.6	9.64	-0.04	0.04	0.0016
Sum of absolute residuals				0.38	
Sum of squared residuals					0.0192

LESSON 3

Squaring or Absolute Value?

Materials: graph paper, rulers
Technology: graphing calculator or computer spreadsheet program
Pacing: 1 class period and homework

Overview

This lesson investigates the properties of the absolute-value functions and their graphs and the quadratic functions and their graphs. It also investigates what happens when you combine two or more absolute-value functions or two or more quadratic functions.

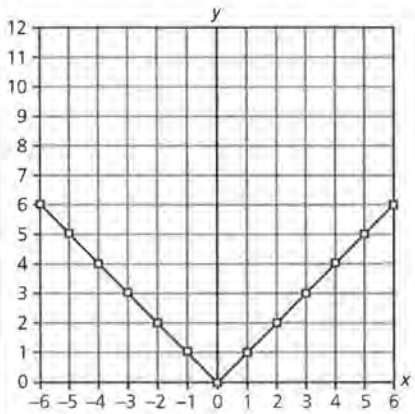
Teaching Notes

The length of time devoted to this lesson will depend on the understanding your students have of these functions and their sums. It is possible that this lesson could be skipped if they have the knowledge already. It is especially important, however, that the students understand what occurs when two or more of each function are combined and what the respective graph looks like.

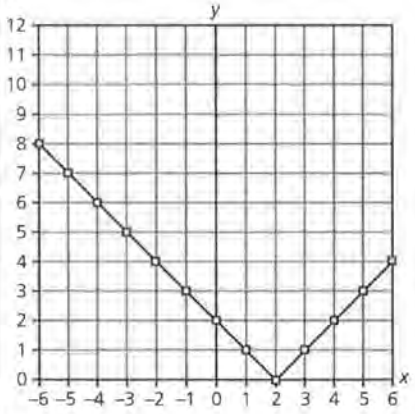
Solution Key

Discussion and Practice

1. The minimum point is (0, 0).

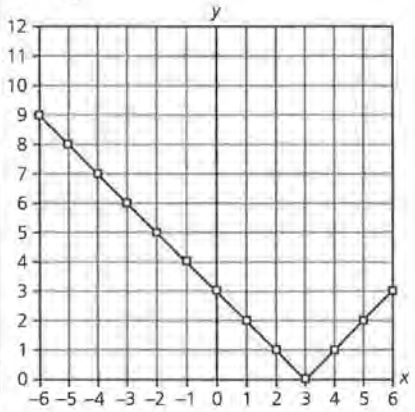


2. The minimum point is (2, 0).

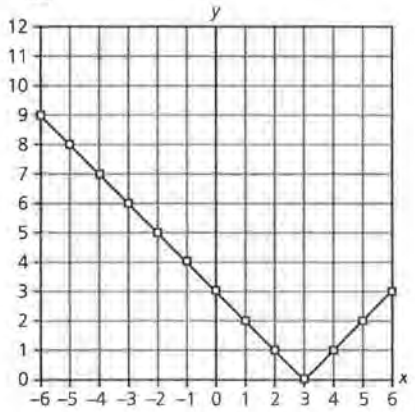


3. The graph of $y = |x - 2|$ is a translation two units to the right of the graph of $y = |x|$.

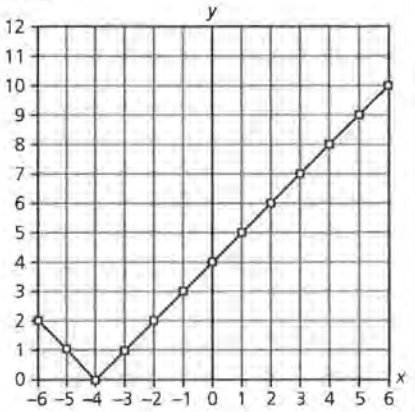
4. a.



b.



c.



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LESSON 3

Squaring or Absolute Value?

If it is important to combine residuals in some manner that will eliminate the negative sign, which method should be used?

Is one method of eliminating the negative sign of residuals easier to work with for all sets of data?

Lesson 2 involved finding the residuals for a given data set and its fitted line and investigated both the absolute value and the square of those residuals as a method of eliminating any negative signs created.

OBJECTIVES

Recognize and describe the graph of quadratic and absolute-value functions.

Recognize what happens when you combine two or more absolute-value functions or two or more quadratic functions.

INVESTIGATE

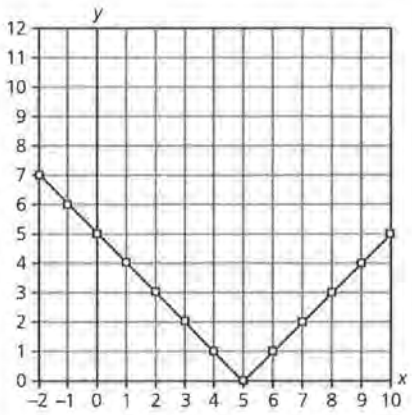
Absolute-Value Functions

An answer to the question in the title can be found by investigating how functions such as $y = (x - 2)^2$ and $y = |x - 2|$ behave and by comparing the results from adding quadratics to the results from adding absolute values.

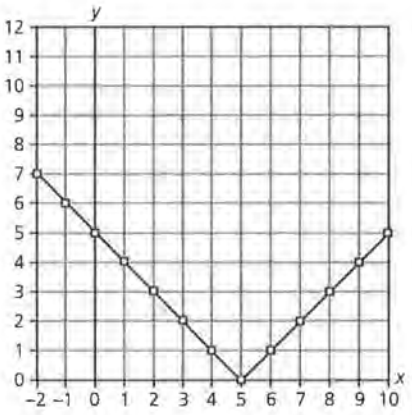
Discussion and Practice

1. Graph the function $y = |x|$. What is the minimum point of the graph?
2. Graph the function $y = |x - 2|$. What is the minimum point of the graph?
3. Write a comparison of the graphs of $y = |x|$ and $y = |x - 2|$.
4. Graph each absolute-value function.
   a. $y = |x - 3|$ b. $y = |3 - x|$ c. $y = |x + 4|$
   d. $y = |x - 5|$ e. $y = |5 - x|$

d.



e.

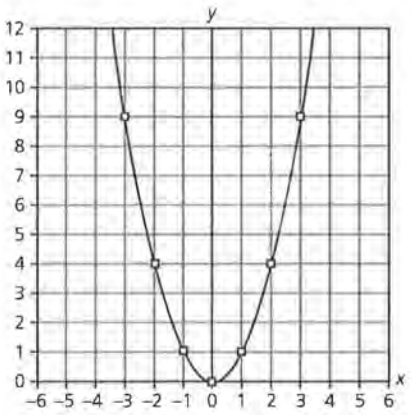


5. **a.** The graph is a translation of the graph of $y = |x|$ a units to the right.

b. The graph is a translation of the graph of $y = |x|$ a units to the right.

c. The graphs are identical.

6. The minimum point is (0, 0).



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5. Use the examples on page 23 for the following.

a. In general, what does the graph of $y = |x - a|$ look like for $a > 0$?

b. In general, what does the graph of $y = |a - x|$ look like for $a > 0$?

c. Write a comparison of the graphs of $y = |x - a|$ and of $y = |a - x|$.

Summary

The graph of the ***absolute-value function,*** of the form $y = |x - a|$, is shaped like a V. The sides of the V are rays, and each has a constant slope, although the slope of one ray is positive and the other is negative. The graph is symmetric around the line $x = a$, which is parallel to the y-axis. This line is called the ***axis of symmetry.***

Quadratic Functions

6. Graph the function $y = x^2$. What is the minimum point of the graph?

7. Graph the function $y = (x - 2)^2$. What is the minimum point of the graph?

8. Write a comparison of the graphs of $y = x^2$ and $y = (x - 2)^2$.

9. Graph each quadratic function.

a. $y = (x - 3)^2$ **b.** $y = (3 - x)^2$ **c.** $y = (x + 4)^2$

d. $y = (x - 5)^2$ **e.** $y = (5 - x)^2$

10. Use the examples above for the following.

a. In general, what does the graph of $y = (x - a)^2$ look like for $a > 0$?

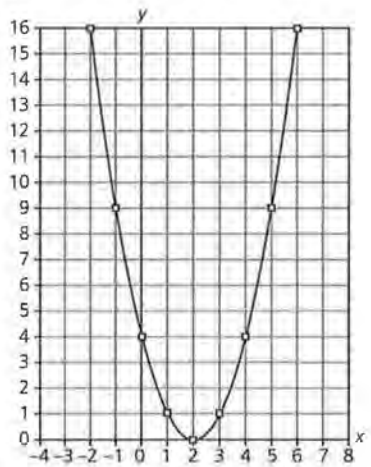
b. In general, what does the graph of $y = (a - x)^2$ look like for $a > 0$?

c. Write a comparison of the graphs of $y = (x - a)^2$ and of $y = (a - x)^2$.

Summary

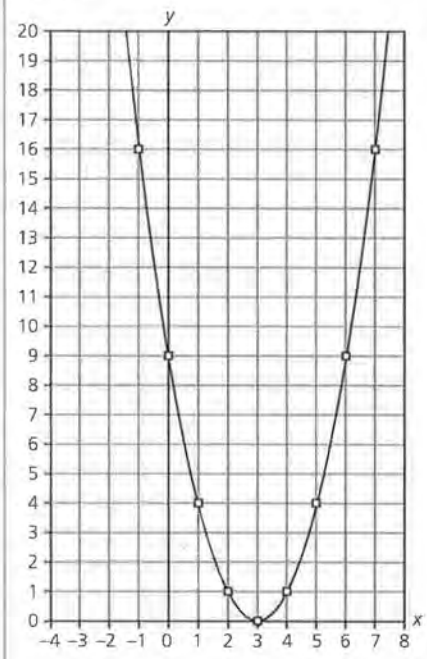
In general, if a is any real number, the equation of the form $y = (x - a)^2$ describes a ***quadratic function.*** The graph of a quadratic equation is called a ***parabola.*** The parabola is generally shaped like a U, and the vertex of this U is called a ***turning***

7. The minimum point is (2, 0).

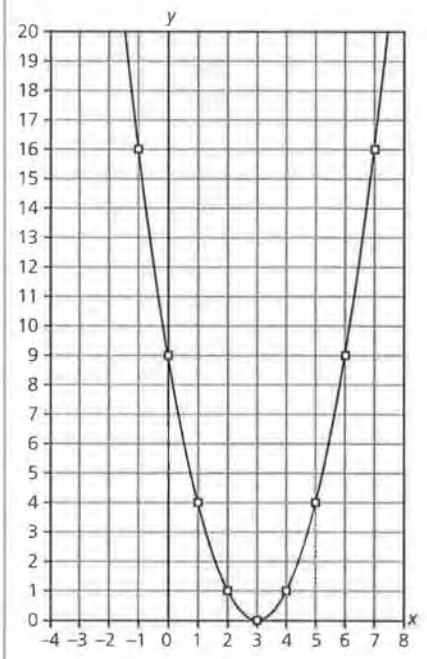


8. The graph of $y = (x - 2)^2$ is a translation two units to the right of the graph of $y = x^2$.

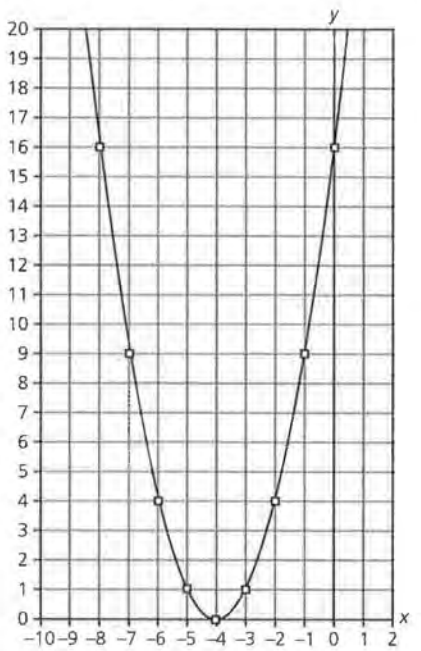
9. a.



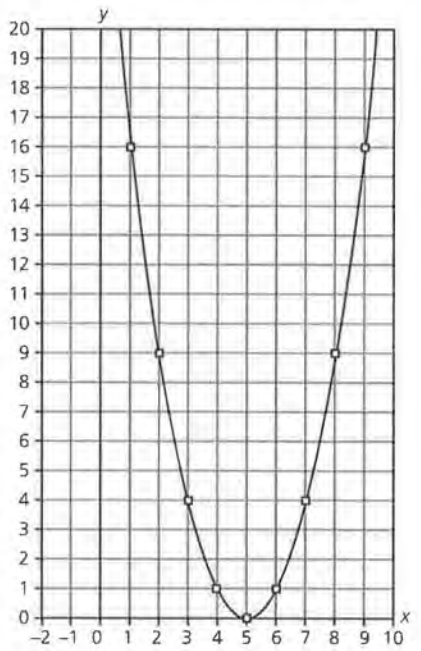
b.



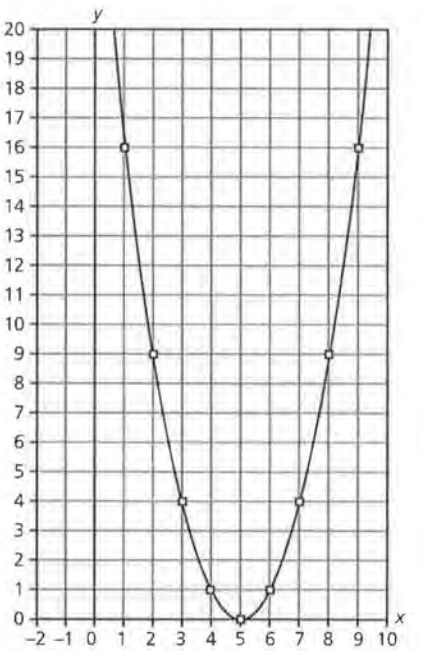
c.



d.



e.



10. a. The graph of $y = (x - a)^2$ is a translation of $y = x^2$ a units to the right.

b. The graph of $y = (a - x)^2$ is a translation of $y = x^2$ a units to the right.

c. The graphs are identical.

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point. If the graph opens up, this point is a ***minimum point.*** If the graph opens down, the point is a ***maximum point.*** The graph is symmetric around the line $x = a$, which is parallel to the y-axis. This line is called the ***axis of symmetry.***

Combining Absolute-Value Functions

In Lesson 2, the sum of the absolute values of the residuals and the sum of the squares of the residuals were found.

To help decide which of these two methods might be better or easier to work with, we will investigate functions that are sums of absolute values and functions that are sums of squares.

11. Consider $y = |5 - x| + |2 - x|$.

- **a.** Describe the graph you think this sum of absolute-value expressions will have.
- **b.** Graph the function. How successfully did you describe the graph?
- **c.** What is the minimum y-value? Describe the x-values that give this minimum.

12. Now, introduce a third absolute-value difference: $y = |5 - x| + |2 - x| + |8 - x|$.

- **a.** Describe the graph you think this sum of absolute-value expressions will have.
- **b.** Graph the function. How successfully did you describe the graph?
- **c.** What is the minimum y-value? Describe the x-values that give this minimum.
- **d.** Comment on this statement: *The graph of a function made up of the sum of a set of absolute values is shaped like the absolute-value function,* $y = |x|$.

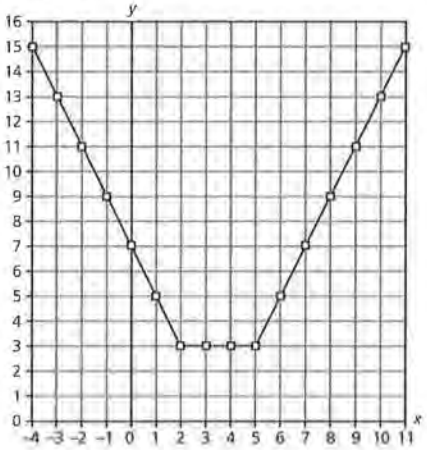
Combining Quadratic Functions

13. Now consider $y = (5 - x)^2 + (2 - x)^2$.

- **a.** Describe the graph you think this sum of squares has.
- **b.** Graph the function. How successfully did you describe the graph?
- **c.** What is the minimum y-value? Describe the x-values that give this minimum.

11. a. Answers will vary. Most students will probably expect the graph to have a V shape.

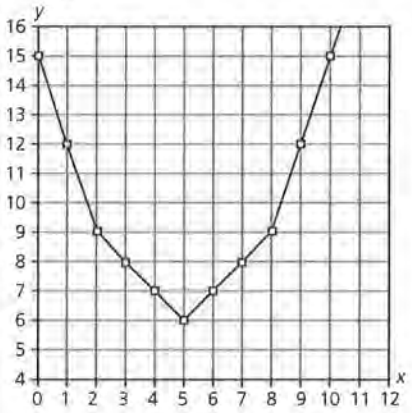
b.



c. The minimum y-value is 3. All x-values between, and including, 2 and 5 have this y-value.

12. a. Most students will again suggest that the graph will be similar to the one just graphed.

b.



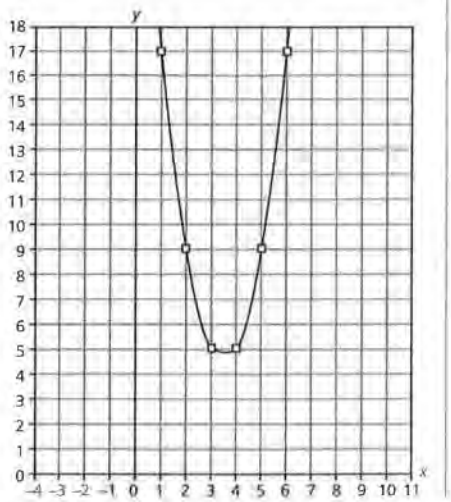
c. The minimum y-value of this graph is 6. The minimum y-value occurs when the x-value is 5.

d. As seen from these examples, the graph of a function that is made up of the sum of absolute values is not a simple V shape.

13. a. Most students will probably think that it forms a parabola.

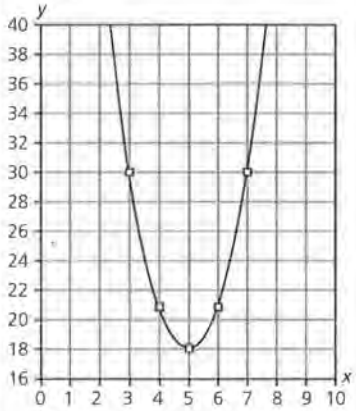
b. See graph at the right.

c. The minimum y-value is 4.5. The x-value at this minimum point is 3.5.



14. a. Most students will probably conjecture the same shape, or a parabola.

b.

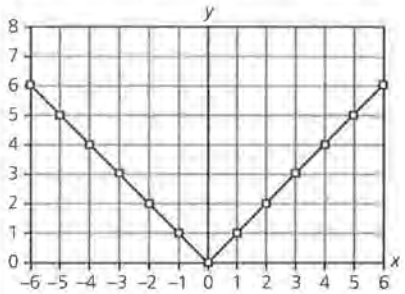


c. The minimum *y*-value is 18. The *x*-value at this minimum point is 5.

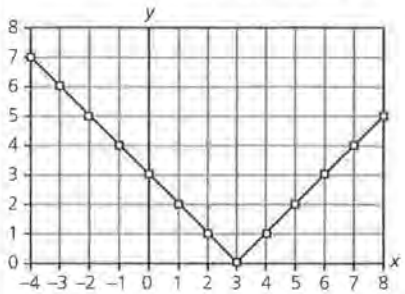
d. From these examples, the statement appears to be correct.

Practice and Applications

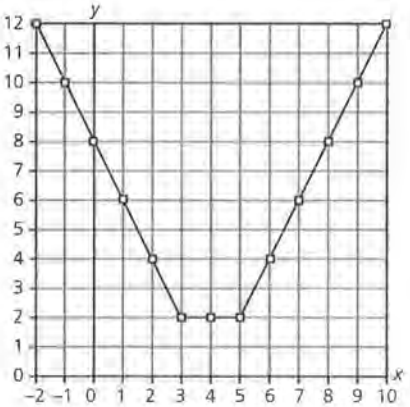
15. a. The minimum point is (0, 0).



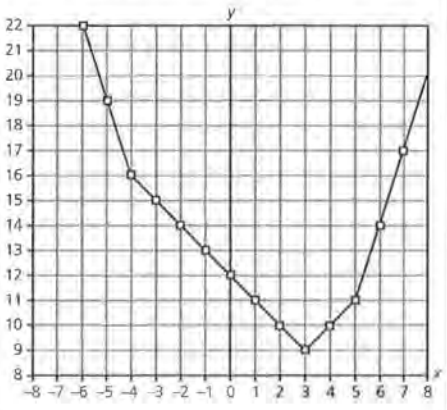
b. The minimum point is (3, 0).



c. The minimum *y*-value is 2, for $3 \le x \le 5$.



d. The minimum point is (3, 9).



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14. What will the graph of the sum of three squared differences look like?

a. Make a conjecture about the shape of the graph $y = (5 - x)^2 + (2 - x)^2 + (8 - x)^2$.

b. Graph the function.

c. What is the minimum *y*-value? Describe the *x*-values that give this minimum.

d. Comment on this statement: *The graph of a function made up of the sum of a set of quadratic expressions is shaped like the graph of a quadratic equation, $y = x^2$.*

Summary

When investigating the absolute-value function, it became difficult to predict what the graph would look like as the number of absolute-value terms increased. The graph of $y = |x|$ is V-shaped, but as more absolute-value terms are added, the shape of the graph changes and is difficult to analyze. The graph of $y = |x|$ has one ordered pair that is a minimum, but as terms are added there may be many ordered pairs with the minimum *y*-value. The graph of a set of squared differences, however, does not cause the same confusion. No matter how many quadratic terms are summed, the graph remains a parabola. Each graph has one and only one *x*-value that corresponds to the minimum *y*-value. The sum of the squares will be a familiar curve, a parabola, which is easy to graph and analyze, and always has one minimum point. Statisticians normally work with a line of best fit that uses squared differences rather than absolute differences. That line is called the ***Least-Squares Regression Line.***

Practice and Applications

Graph each equation and determine its minimum point(s).

15. a. $y = |x|$ **b.** $y = |3 - x|$

c. $y = |3 - x| + |5 - x|$

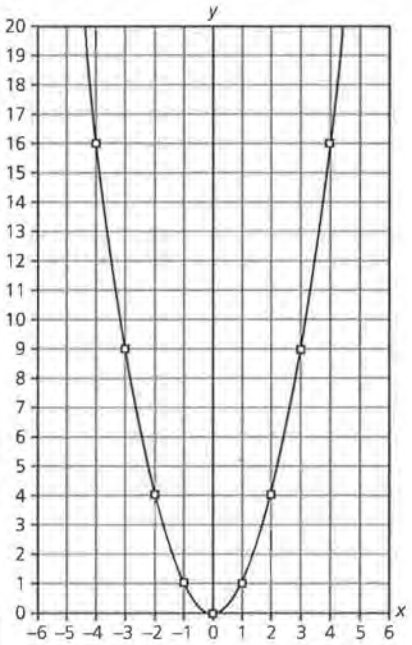
d. $y = |3 - x| + |5 - x| + |4 + x|$

16. a. $y = x^2$ **b.** $y = (3 - x)^2$

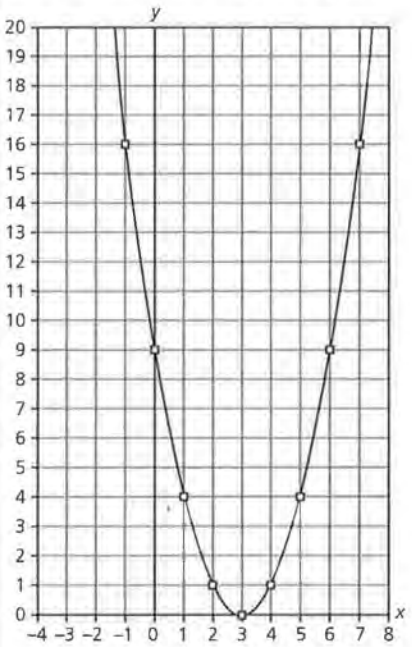
c. $y = (3 - x)^2 + (5 - x)^2$

d. $y = (3 - x)^2 + (5 - x)^2 + (4 + x)^2$

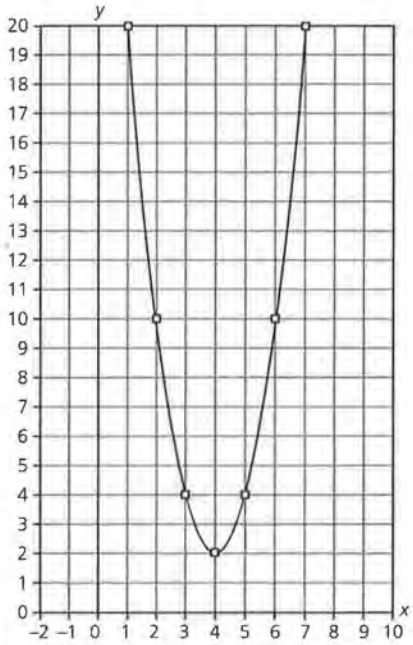
16. a. The minimum point is (0, 0).



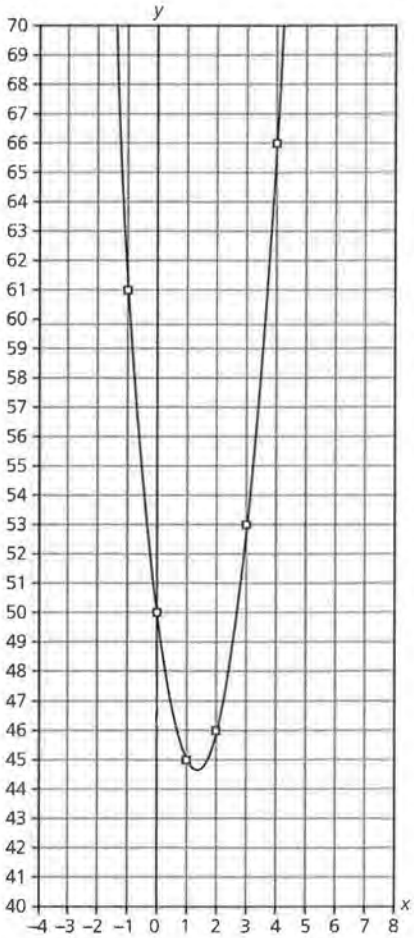
b. The minimum point is (3, 0).



c. The minimum point is (4, 2).



d. The minimum point is $(\frac{4}{3}, 44\frac{2}{3})$.



LESSON 4

Finding the *Best* Slope

Materials: graph paper, rulers, *Activity Sheet 3*
Technology: graphing calculator or computer spreadsheet program
Pacing: 1 class period

Overview

In Lesson 3, the method of minimizing the sum of squared residuals was accepted as a means of determining a *best* line. Trying a variety of equations eventually gives you a line that has a small sum of squared residuals. This lesson investigates the relationship between the slope of a line and the sum of the squared residuals of the line; that is, students will find the slope that minimizes the sum of squared residuals.

Teaching Notes

This lesson requires the use of technology. The calculations necessary to make intelligent decisions would become so tedious that students' understanding would be impaired. The lesson is best suited for a whole-class lesson because you would be able to collect a great deal of data quickly by combining the data from individual class members.

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LESSON 4

Finding the *Best* Slope

How can you know you really do have the *best* line?

How can you be sure that different people investigating a problem will come up with exactly the same results?

How can the *best* line be found without trying all the possibilities?

Can the slope be found that minimizes the sum of squared residuals?

In the last lesson, the method of minimizing the sum of squared residuals was accepted as a means of determining a *best* line. Trying a variety of equations eventually gives you a line that has a small sum of squared residuals.

OBJECTIVE

Find the slope of a line that minimizes the sum of the squared residuals.

INVESTIGATE

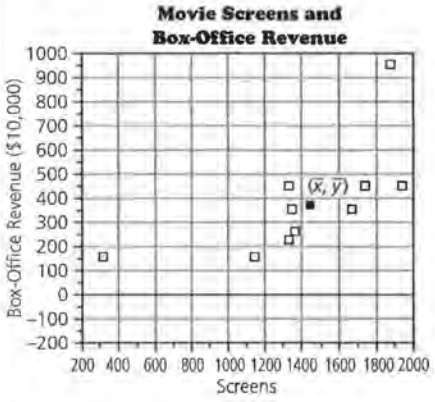
This lesson investigates the relationship between the slope of a line and the sum of the squared residuals of the line; that is, you will find the slope that minimizes the sum of squared residuals.

Solution Key

Discussion and Practice

1. a. $(\bar{x}, \bar{y}) = (1418.2, 375.1)$

b.



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There are very sophisticated methods that can be employed to determine this line, but the method introduced here produces the same result. The equation of a line is determined by a point on the line and its slope, and these values vary from line to line. One way to determine the slope of a line that minimizes the sum of squared residuals is to fix a point and vary the slope. The procedure will be to guess that the best line will most likely contain the center point of the data. Based on that assumption, identify the center point and consider lines with different slopes that would pass through that point. From the sum of the squared residuals for each of these lines, it will be possible to find the slope that yields the minimum squared residuals.

In Lesson 5, you will use that slope and vary the *y*-intercept. From the sum of the squared residuals for each of these lines, it will be possible to determine the *y*-intercept of the line that has the minimum sum of squared residuals.

Discussion and Practice

The first step is to decide which point should be fixed. The box-office revenue for any movie could be estimated without knowing the number of screens by using $\bar{y}$, the mean box-office revenue of the movies on the list. Likewise, the mean number for all the screens, that is, $\bar{x}$, can serve as an estimate for the number of screens for any movie being studied.

Since the mean is a value that can be used to summarize the center of a distribution, it seems reasonable to use $(\bar{x}, \bar{y})$ as the fixed point. This point is called the ***centroid***. With this point fixed, you can then draw lines with different slopes through this point to find the slope that minimizes the sum of the squared residuals.

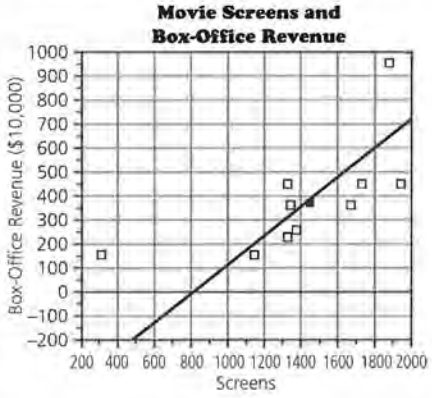
1. Use the data on the top ten films from Lesson 2.
   a. Determine the mean number of screens and the mean reported box-office revenue. Call this point $(\bar{x}, \bar{y})$.
   b. Locate the point $(\bar{x}, \bar{y})$ on the scatter plot of the box-office revenue on a clean copy of *Activity Sheet 3*.

There are many lines that pass through the point $(\bar{x}, \bar{y})$. Investigating some of these will help determine the relationship between the slope and the sum of the squared residuals.

c. For line ***L1,*** another possible point is (800, 100), which gives a slope of 0.445 and an equation $y = 375.1 + 0.445(x - 1418.2)$.

For the line ***L2,*** a possible point is (600, 200), which gives a slope of 0.214 and an equation $y = 375.1 + 0.214(x - 1418.2)$.

d. Answers will vary. One possible line is $y = 375.1 + 0.589(x - 1418.2)$.



2. For the lines given and the one proposed in part d above:

Slope	Sum of Squared Differences
0.445	325,589.14
0.214	326,657.25
0.589	431,087.28

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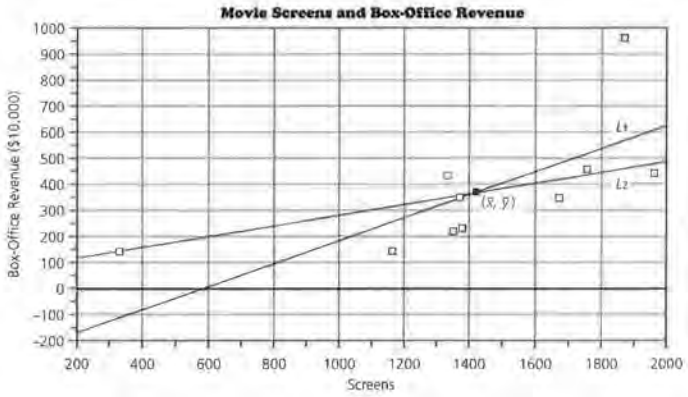

c. A useful way to write an equation given a point (x_1, y_1) and the slope, m, is $y = y_1 + m(x - x_1)$. Use the centroid and another point on each line from the graph above to calculate the slope. Then write an equation for each line in the form $y = y_1 + m(x - x_1)$.

d. Draw a line through the point $(\bar{x}, \bar{y})$ that you think summarizes the data fairly well. Find the slope of the line and then write an equation of your line in the form $y = \bar{y} + m(x - \bar{x})$. Compare your equation with those of your classmates. How are the equations similar? How are the equations different?

The next step is to find the sum of squared residuals for each line.

2. Use a spreadsheet or calculator to find the sum of squared residuals. Refer to the example that follows. Record the slope of your line and the sum.

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3. **a.** B1 contains the slope of the line. A3 is the number of screens for the first movie.

b. They are the coordinates of the point in all of the lines, the centroid of the data.

c. The predicted value of the revenue for the number of screens

d. The square of the difference between the predicted value and the actual value

Option A: Spreadsheet

Enter the slope you are using in B1. Type the equation in C3, the rule for the squared difference in D3, and fill down both columns.

	A	B	C	D
1	Slope=			
2	Screens	Box-Office Revenue	Predicted Revenue	Square of Residual
3	1878	964	=375.1+$B$1*(A3–1418.2)	=(B3–C3)^2
4	1753	460		
5	1963	448		
6	1329	436		
7	1363	353		
8	1679	352		
9	1383	230		
10	1346	212		
11	325	150		
12	1163	146		
			Sum =	=Sum(D3:D12)

3. Use the spreadsheet above to answer the following.

a. The formula in cell C3 uses cell locations B1 and A3. What data are stored in each of these cells?

b. What do the values 1418.2 and 375.1 represent in the formula in C3?

c. What does the formula in C3 calculate?

d. What does the formula in D3 calculate?

4. **a.** Y1 = 375.1 + 0.4(x – 1418.2)

b. The predicted value for the value in L1(3)

c. The square of the difference between L2(2) and L3(2)

d. $\Sigma(\text{residuals})^2 = 309{,}341.8$

5. Answers will vary.

6. **a.** Use the equation $y = 375.1 + 0.22(L1 - 1418.2)$.

b. $y = 375.1 + 0.3382(x - 1418.2)$

c. The value of 300,000 is the squared residual for 10 screens, so the average squared difference is 30,000. This means that on the average you would expect the difference between the prediction and the actual value to be 173.21, or $1,732,100.

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Option B: Calculator

Define L3 using quotation marks and your equation.

Define L4 as " (L2 – L3)^2 ".

Screens	Box Office	L3	L4
1878	964		
1753	460		
1963	448		
1329	436		
1363	353		
1679	352		
1383	230		
1346	212		
325	150		
1163	146		
L3 =			

4. Use the slope 0.4 and the point (1418.2, 375.1).

a. What equation do you enter in Y1?

b. What does the value in L3(3) represent?

c. What does the value in L4(2) represent?

d. Find the sum of squared residuals.

At this point you have the equation for one line through the point $(\bar{x}, \bar{y})$. The goal is to find a slope that minimizes the sum of squared residuals. Try several more lines, each line with a different slope, but all passing through $(\bar{x}, \bar{y})$.

5. Record the slopes and sum of squared residuals found by the rest of the class.

6. Data collected by three students in a mathematics class are at the right.

Slope	$\Sigma(\text{residuals})^2$
0.22	323,975
0.45	327,886
0.26	309,714

a. Describe how the value of 323,975 was obtained.

b. Write an equation of the line used to obtain 300,000 for $\Sigma(\text{residuals})^2$.

c. Use the value 300,000 to find the value of the root mean squared error. What does this value tell you about predicting the movie revenue from the number of screens?

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7. Plot the three ordered pairs (slope, $\Sigma(\text{residuals})^2$) from Problem 6.
   a. Plot the ordered pairs (slope, $\Sigma(\text{residuals})^2$) you found from the lines you and other students in class used. What pattern do you see in the scatter plot?
   b. Draw a smooth curve through the ordered pairs on the graph. What kind of equation might be used to describe this graph?
   c. Find the x-coordinate of the point that has the least y-coordinate. Write the coordinates of this point.
   d. Describe what the x-coordinate and y-coordinate of this point represent.

Summary

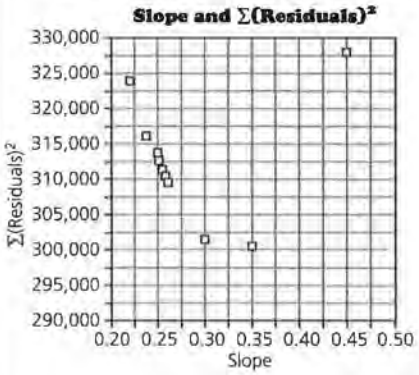
The overall goal has been to find the equation of the line that *best* fits the data.

In this lesson, you found a slope that minimized the sum of squared residuals starting with the fixed point $(\bar{x}, \bar{y})$. The slope from any line through this point generated its own $\Sigma(\text{residuals})^2$. The ordered pairs (slope, $\Sigma(\text{residuals})^2$) formed a parabola where the x-coordinate of the minimum point represented the slope of the line that had the least $\Sigma(\text{residuals})^2$.

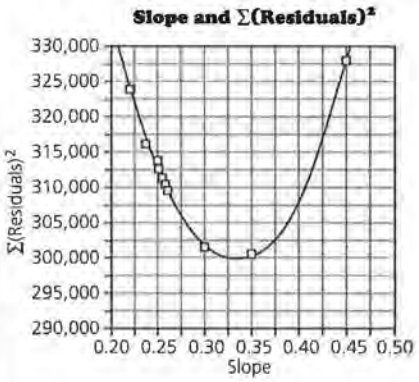
Practice and Applications

8. Write a paragraph discussing how to find an estimate for the value of the slope that minimizes the sum of squared residuals.
9. Use the data on BMX dirt-bike racing found at the end of Lesson 1 to find the value of the slope that minimizes the sum of squared residuals, starting with the point $(\bar{x}, \bar{y})$.

7. a. Answers will vary. A sample is given below. The points appear to lie on a curve, possibly a parabola.



b. The equation is a quadratic equation.



c. The minimum appears to be at (0.33, 300,000).

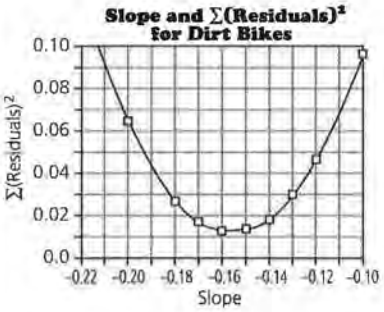
d. The y-coordinate represents the least-squared-error sum, so the x-coordinate represents the slope that gives the least sum of squared residuals.

Practice and Applications

8. Student paragraphs should include the idea that you need to find a number of slope-residual pairs and plot them. From the plot, sketch a parabola and use this to estimate the minimum point. The x-coordinate of the minimum is the slope of the line of best fit through the point $(\bar{x}, \bar{y})$.

9. For the data set, the fixed point $(\bar{x}, \bar{y})$ is (21.5, 10). Equations will be of the form $y = 10 + m(x - 21.5)$.

Slope	Sum of Squared Residuals
–0.10	0.0986
–0.20	0.0656
–0.15	0.01335
–0.12	0.048
–0.13	0.03095
–0.14	0.0194
–0.16	0.0128
–0.17	0.01775
–0.18	0.0282



It appears the minimum point has coordinates (–0.156, 0.0124) and a predicted equation is $y = 10 - 0.156(x - 21.5)$.

LESSON 5

Finding the *Best* Intercept

Materials: graph paper, rulers, *Activity Sheets 3* and *4*
Technology: graphing calculator or computer spreadsheet program
Pacing: 1 class period

Overview

The collective goals in Lesson 4 and in this lesson are to find the slope and the intercept of the line that minimizes the sum of squared residuals. In Lesson 4, by drawing several lines through the point $(\bar{x}, \bar{y})$, students found a slope that gave the least sum of squared residuals. Remind students that we made the assumption that the line had to pass through the point with coordinates $(\bar{x}, \bar{y})$, the mean of the xs and the mean of the ys. In this lesson, the slope will be held constant and the point varied.

Note: One should hold slope and the x-coordinate, $\bar{x}$, constant and vary the value of the y-coordinate. However, this would have the same effect as varying the y-intercept. Since it is more convenient to vary the y-intercept, that is the course of action in this lesson.

The goal is to find the point that minimizes the sum of squared residuals.

Teaching Notes

This lesson requires the use of technology. Again, the calculations necessary to make intelligent decisions would become so tedious that students' understanding would be impaired. The lesson is best suited for a whole-class lesson because you would be able to collect a great deal of data quickly by combining the data from individual class members.

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LESSON 5

Finding the *Best* Intercept

What do you think will happen if you keep the slope the same but change the point that the line passes through?

OBJECTIVE

Investigate the relationship between the intercept and the sum of squared residuals.

INVESTIGATE

The goals in Lesson 4 and in this lesson are to find the slope and the y-intercept of the line that minimizes the sum of squared residuals. By drawing several lines through the point (1418.2, 375.1) in Lesson 4, you found that a slope of approximately 0.33 gave the least sum of squared residuals. But remember, we made the assumption that the line had to pass through the point with coordinates $(\bar{x}, \bar{y})$, the mean of the number of screens and the mean of the box-office revenue. In this lesson, the slope will be held constant and the point will be varied. Remember, the goal is to find the point that minimizes the sum of squared residuals.

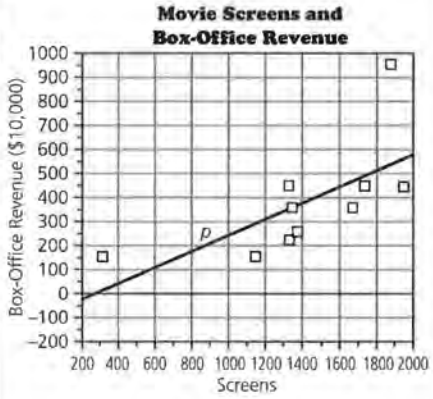
Discussion and Practice

1. The scatter plot on page 34 is reproduced on *Activity Sheet 3*. Use a clean copy of the activity sheet, and draw the line p that passes through the point (1418.2, 375.1) and has a slope of 0.33.

Solution Key

Discussion and Practice

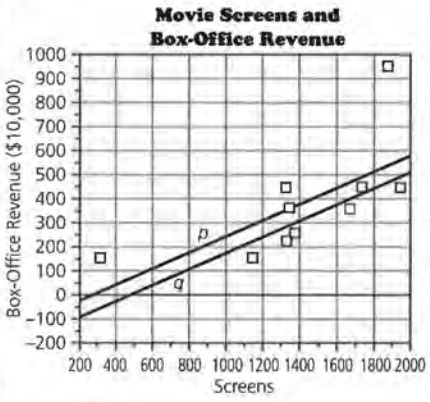
1.



a. $y = 375.1 + 0.33(x - 1418.2)$

b. $y = 0.33x - 92.906$; the y-intercept is -92.906.

2.



a. The slope of line q is 0.33, since it is parallel to line p.

b. $y = 370 + 0.33(x - 1600)$

c. $y = 0.33x - 158$; the y-intercept is -158.

3. **a.** The slope will remain the same and the y-intercept will change.

b. The value of b will change.

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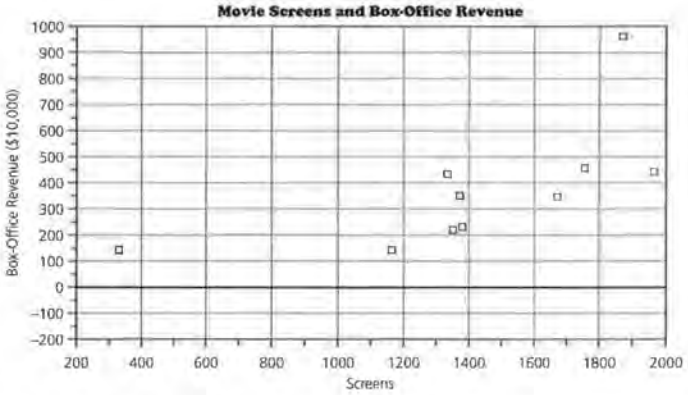

a. Write the equation of line p in the form $y = y_1 + m(x - x_1)$, where m is the slope and (x_1, y_1) is any point on the line.

b. Write the equation of line p in the form $y = mx + b$, where b is the y-coordinate of the y-intercept and m is the slope. What is the y-intercept of the line with slope equal to 0.33 that passes through $(\bar{x}, \bar{y})$?

2. On your scatter plot from Problem 1, draw a line q that is parallel to line p and passes through the point (1600, 370).

a. What is the slope of line q? How did you find your answer?

b. Write the equation of line q in the form $y = y_1 + m(x - x_1)$.

c. Write the equation of line q in the form $y = mx + b$. What is the y-intercept of this line?

3. Suppose you choose another point and draw a line parallel to line p.

a. What effect will this have on the slope and the y-intercept?

b. What effect does changing the point have on the equation of line p when the equation is written in the $y = mx + b$ form?

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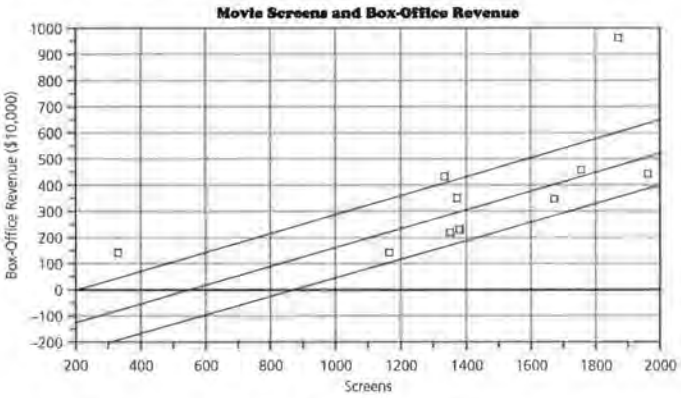
4. The lines below can be considered a *family* of lines. Describe the similarities and differences.



The goal of this lesson is to draw a family of lines that all have the same slope, 0.33, and to find the y-intercept of the line that gives the least value for the sum of squared residuals. Just as before, use a spreadsheet or graphing calculator to investigate the change in the sum of squared residuals as different lines are drawn on the plot of (number of screens, box-office revenue). Remember this important point: Holding the slope constant and changing from the centroid to other points for the line to pass through also changes the y-intercept.

5. Find the sum of squared residuals for line *q* from Problem 2. If you use a spreadsheet, you have to enter the y-intercept of the line, so you must write the equation of the line in slope-intercept ($y = mx + b$) form. Record your results in a table similar to one of those on page 36.

4. The slopes of all of the lines are the same since they are parallel. The *y*-intercepts are different.

5. For $y = 0.33x - 158$, the sum of squared residuals is 342,266.74. See table below.

Screens	Box-Office Revenue	Predicted Revenue	Square of Residual
1878	964	461.74	252,265
1753	460	420.49	1,561
1963	448	489.79	1,746.4
1329	436	280.57	24,158
1363	353	291.79	3,746.7
1679	352	396.07	1,942.2
1383	230	298.39	4,677.2
1346	212	286.18	5,502.7
325	150	−50.75	40,301
1163	146	225.79	6,366.4

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Option A: Spreadsheet
Enter the intercept you are using in B1, the equation in C3, and the rule for the squared difference in D3. Fill down columns C and D.

	A	B	C	D
1	*y*-Intercept=			
2	Screens	Box-Office Revenue	Predicted Revenue	Square of Residual
3	1878	964	=0.33*A3+$B$1	=(B3–C3)^2
4	1753	460		
5	1963	448		
6	1329	436		
7	1363	353		
8	1679	352		
9	1383	230		
10	1346	212		
11	325	150		
12	1163	146		
13			Sum =	=Sum(D3:D12)

Option B: Calculator
Type the equation you are using in Y1.

Define L3 as Y1(L1) and L4 as $(L2 - L3)^2$.

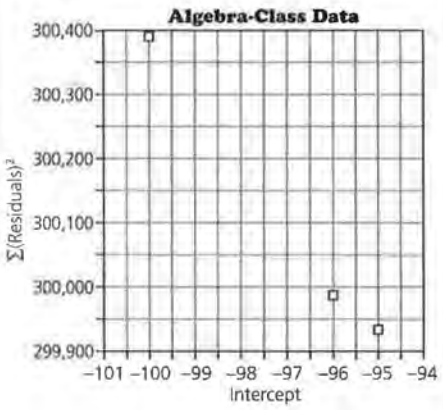
L1	L2	L3
1878	964	
1753	460	
1963	448	
1329	436	
1363	353	
1679	352	
1383	230	
1346	212	
325	150	
1163	146	
L3 = Y1(L1)		

6.

Slope	Intercept	Sum of Squared Residuals
0.33	−92.906	299,894.45
0.33	−158	342,266.74
0.33	−105	301,357.10
0.33	−90	299,978.90
0.33	−85	300,519.50

7. **a.** $y = 0.33x - 96$; if you subtract 96 from one third the number of screens, the box-office revenue will be \$10,000 times the result.

b.



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6. Draw at least two more lines parallel to line p on your scatter plot from Problems 1 and 2. Write an equation of each line in the form $y = 0.33x + b$ and find the sum of the squared residuals. Record your results in a table like the following, or use *Activity Sheet 4*.

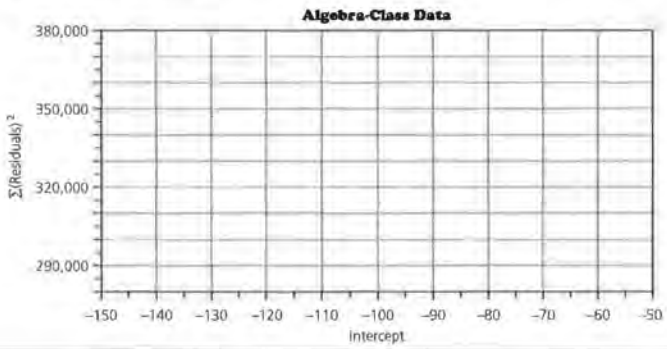
Slope	Point	Intercept	Sum of Squared Residuals
0.33	(1679, 352)	_____	_____
0.33	_____	_____	_____
0.33	_____	_____	_____
0.33	_____	_____	_____
0.33	_____	_____	_____

7. Data collected by several students in an algebra class are below.

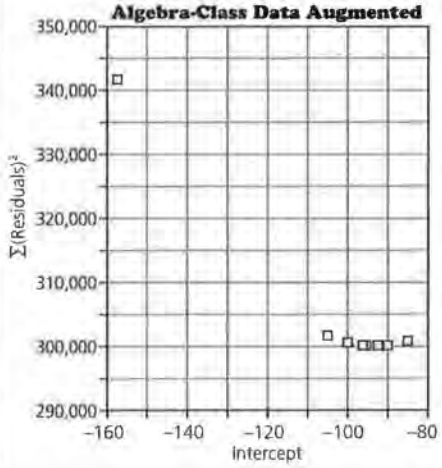
Slope	Intercept	Σ(Residuals)²
0.33	−100	300,398
0.33	−96	299,990
0.33	−95	299,938

a. Write an equation of the line that gave $\Sigma(\text{residuals})^2 = 299{,}990$. What does this equation tell you about the number of screens and box-office revenue?

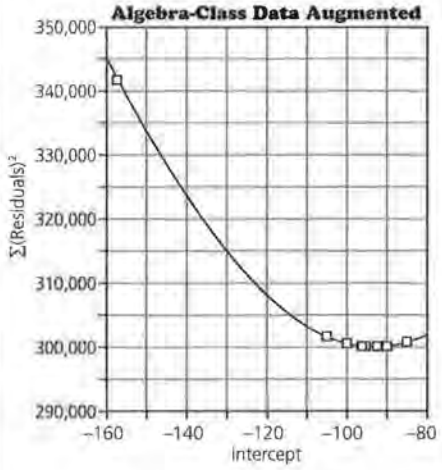
b. Use a grid like the one below, reproduced on *Activity Sheet 4*, or your graphing calculator and plot the three ordered pairs (intercept, $\Sigma(\text{residuals})^2$) from the table shown in this problem.



c. Answers will vary. The graph again seems to be a parabola.



d. The equation is a quadratic equation.



e. The minimum point appears to be about (–93, 299,900).

f. This means that the equation $y = 0.33x - 93$ gives what appears to be the least sum of squared residuals, which is about 299,900. The answer to Problem 1b is –92.9, which is very close to –93.

8. Student paragraphs should include the idea of trying a number of lines and plotting (intercept, residuals) pairs. From the plot, estimate a parabola and find the minimum point. Use the minimum point to estimate the minimum intercept.

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c. On the plot, add the ordered pairs (intercept, $\Sigma(\text{residuals})^2$) you found from the lines you drew. Add ordered pairs from your classmates. Describe any patterns you observe in the scatter plot.

d. Draw a smooth curve through the ordered pairs on the graph. What kind of equation might be used to describe this curve?

e. From the scatter plot, determine the x-coordinate of the point that has the least y-coordinate. Write the coordinates of this point.

f. Describe what the x-coordinate and the y-coordinate of this point represent.

8. In this lesson, you fixed the slope at 0.33 and then drew lines with this slope. Write a paragraph discussing how to find the value of the intercept that minimizes the sum of squared residuals for this fixed slope.

Summary

In this lesson the slope was fixed and points (intercept, $\Sigma(\text{residuals})^2$) were generated. The plot of these points was a parabola with the point that gave the smallest sum of squared residuals at the minimum point, $(\bar{x}, \bar{y})$, the centroid.

Practice and Applications

9. Use the data on BMX dirt-bike racing found at the end of Lesson 1 to find the value of the y-intercept that minimizes the sum of squared residuals. Use the value of the slope found in Problem 9 of Lesson 4.

Attention: Even though varying the slope while passing the line through the centroid led to the discovery of a slope that created the least sum of squared residuals and varying the point the line was passing through while holding the slope constant created the least sum of squared residuals, care must be taken to not assume that the *best* line will be the one in which we do those changes simultaneously. Lesson 6 will consider the effect of making those changes simultaneously and how that might be done.

Practice and Applications

9. From the last lesson, the slope is –0.156.

Intercept	Sum of Squared Residuals
13.25	0.10804
13.3	0.03504
13.35	0.01204
13.4	0.03904
13.45	0.11604

The y-intercept that appears to give the minimum sum of squared residuals as about 13.35.

LESSON 6

The *Best* Slope and Intercept

Materials: graph paper, rulers, *Activity Sheets 3* and *5*
Technology: graphing calculator or computer spreadsheet program
Pacing: 1 class period

Overview

Recall that the goal of Lessons 4 and 5 was to find the slope and the intercept of the best line that summarizes the data and can be used to make predictions. This line was defined to be the line minimizing the sum of squared residuals. To accomplish this goal, the centroid, $(\bar{x}, \bar{y})$ was fixed, and it was found that a slope of 0.33 minimized the sum of squared residuals. Next, the slope was fixed at 0.33 and the intercept varied; then students found that the intercept –93 minimized the sum of squared residuals. The equation of the line was $y = 0.33x - 93$, and it contained the point $(\bar{x}, \bar{y})$. Is this the best line? To determine the answer to this question, we must vary both the slope and the intercept.

Teaching Notes

This lesson requires the use of technology. The calculations necessary to make intelligent decisions would become so tedious that students' understanding would be impaired. The lesson is best suited for a whole-class lesson because you would be able to collect a great deal of data quickly by combining the data from individual class members. If you want to help the students understand and appreciate the figure of the paraboloid, it might be helpful to use Tinker Toys to indicate the axes and help them to visualize the third dimension. The x-axis represents the slope, the y-axis represents the intercept, and the z-axis represents the sum of the squared residuals. The discussion of the paraboloid could be skipped and the rest of the lesson covered without a loss of continuity.

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LESSON 6

The *Best* Slope and Intercept

What happens to the sum of squared residuals if *both* the slope and intercept are varied?

INVESTIGATE

Recall that the goal was to find the slope and the intercept of the line that best summarizes the data and that can best be used to make predictions. This line was defined to be the line minimizing the sum of squared residuals. To accomplish this goal, the centroid, $(\bar{x}, \bar{y})$ having the value (1418.2, 375.1) was fixed. It was found that a slope of 0.33 minimized the sum of squared residuals. Next, the slope was fixed at 0.33 and the point was varied. In this case, it was found that a *y*-intercept of –93 minimized the sum of squared residuals. The equation of the line was $y = 0.33x - 93$, and it contained the point (1418.2, 375.1). Is this the best line?

The picture that follows represents the families of curves that occur if both the slope and intercept are varied. It is a *paraboloid;* the lowest point of the paraboloid is the point that has the minimum sum of squared residuals.

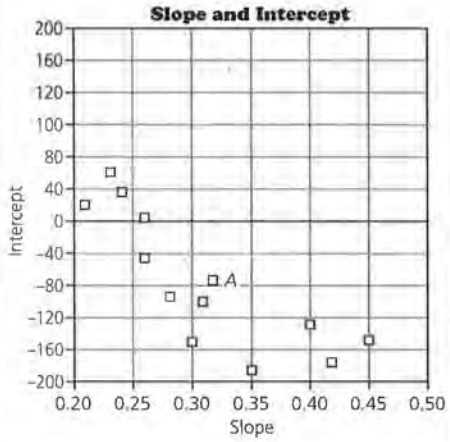
OBJECTIVE

Investigate how the sum of squared residuals depends jointly on the slope and the *y*-intercept.

Solution Key

Discussion and Practice

1. Answers will vary; sample:



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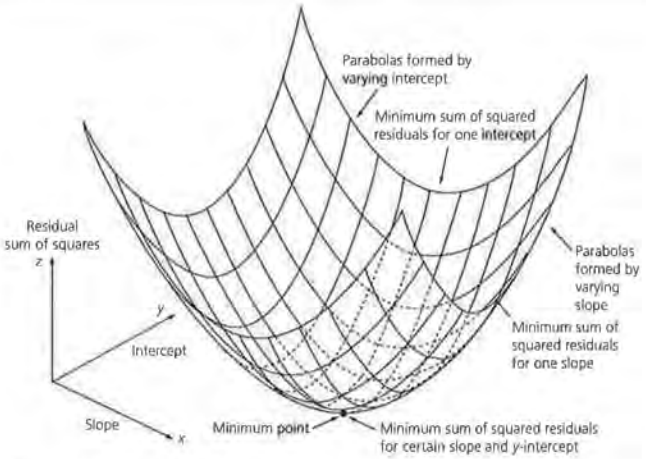
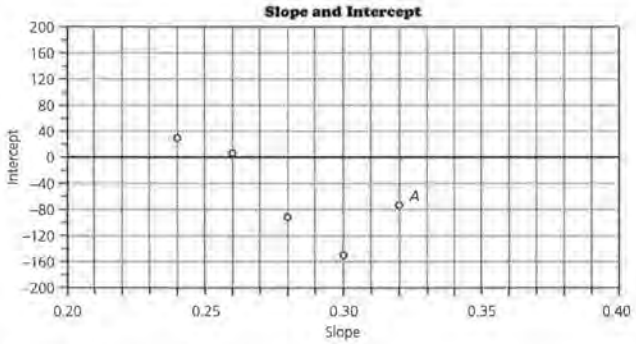


Discussion and Practice

In Lessons 2, 4, and 5, you collected data relating the sum of squared residuals to the slopes and intercepts of several lines. A sample of these data is listed at the right.

Slope	Intercept	Σ(Residuals)²
0.30	−150	401,025
0.28	−93	355,358
0.26	6	309,713
0.32	−78	300,117
0.24	34	316,058

1. The plot below shows the ordered pairs (slope, intercept) from these data. On *Activity Sheet 5*, plot at least eight more points from the data collected in the last lessons. Include the point from Lesson 5, Problem 7e.



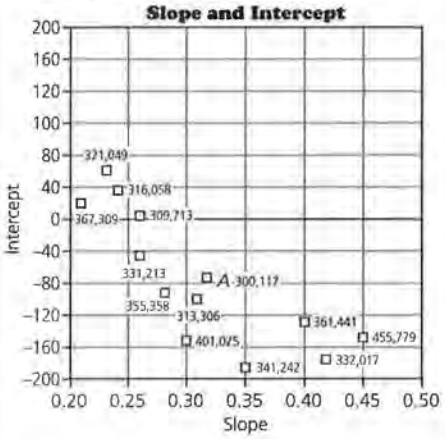
a. What does point *A* represent?

a. Point *A* represents the line with slope 0.32 and intercept –78.

b. $y = 0.32x - 78$

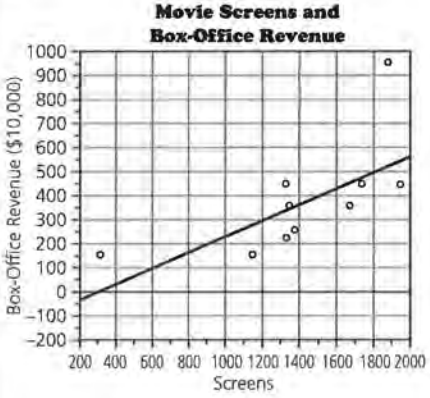
c. Answers will vary. The line $y = 0.33x - 93$ has a smaller sum of squared residuals (about 299,900).

d. Answers will vary. Sample from the graph above:



e. $y = 0.33x - 93$

2.



a. This is the line where the square of the vertical distance from the line to each point is the minimum.

b. This line can be used to estimate the box-office revenue if you know the number of screens.

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b. What is an equation of the line that generated point *A*?

c. Can you find a point where $\Sigma(\text{residuals})^2$ is less than the $\Sigma(\text{residuals})^2$ for point *A*? If so, what is an equation of the line that generated the point?

d. For each point, write the $\Sigma(\text{residuals})^2$ that you collected for the line with the given slope and *y*-intercept.

e. Find the point that has the least $\Sigma(\text{residuals})^2$. What is an equation of the line that generated this point?

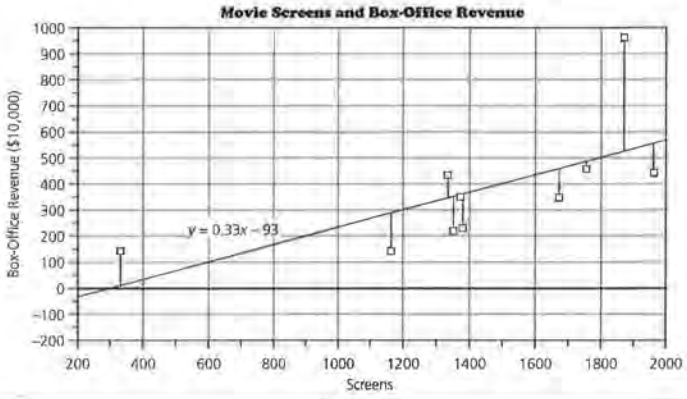
2. Use a clean copy of *Activity Sheet 3* to plot the number of screens and box-office revenue and graph the line whose equation you found in Problem 1e.

a. What does this line represent?

b. How does the line help you to summarize the data?

Summary

In this lesson, the slope and the intercept were varied simultaneously, and it was determined that the minimum point of the paraboloid occurred at the point with the same slope and intercept found in previous lessons. This assures us that we have found the line which minimizes the sum of the squared residuals. Statisticians refer to this line as the ***least-squares line.*** The least-squares line is the line that minimizes the sum of squared residuals, as in the diagram below. This line can be used as a *best* line to summarize data that appear to be linear.



Practice and Applications

3. Each square is created using the length of the residual of the point as the side of the square. The picture provides a visual picture of all the squares. If we were to move the line, the size of the various squares would increase or decrease depending on the location of that line. When we arrive at the line that makes the sum of the areas of the squares its least value, this line is the least-squares regression line.

4. For the bike data, the least-squares line is $y = -0.156x + 13.35$.

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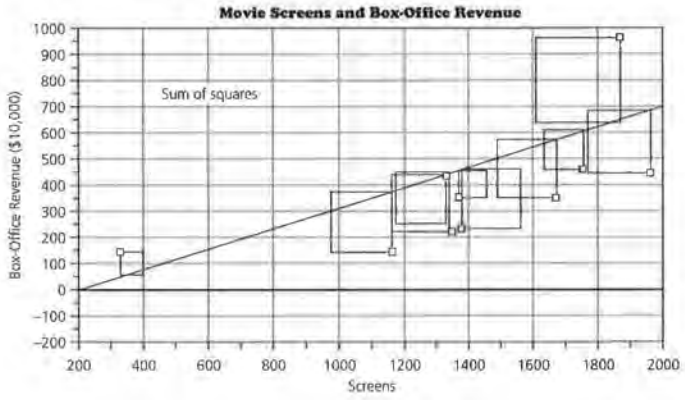
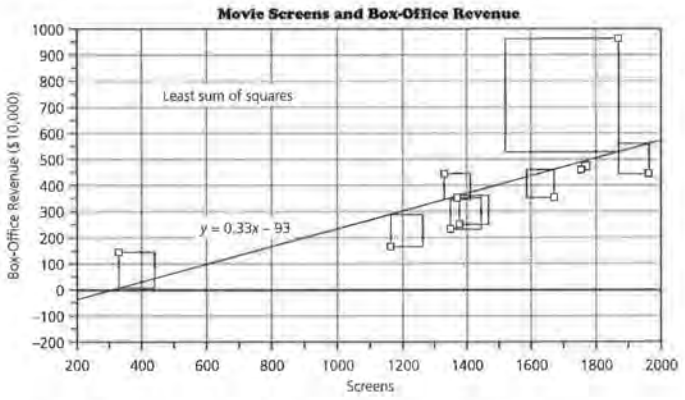
Practice and Applications

3. Consider the diagrams below. Explain the squares and how they relate to finding a least-squares regression line.





4. Find the least-squares regression line for the BMX dirt-bike data from Lesson 1.

LESSON 7

Quadratic Functions and Their Graphs

Materials: graph paper, rulers, *Activity Sheet 6*
Technology: graphing calculator or computer spreadsheet program
Pacing: 1–2 class periods and homework

Overview

The quadratic function created by summing the squares of the residuals was used to determine the *best* line because the graph would always have a minimum point. The goal of previous work was to find the minimum point of the curve graphically and numerically. In this lesson, algebra will be used to find the minimum point. Every straight line can be described with many equivalent equations, and different forms of the equation reveal different characteristics of the line. This lesson investigates the corresponding issues for quadratic functions.

Teaching Notes

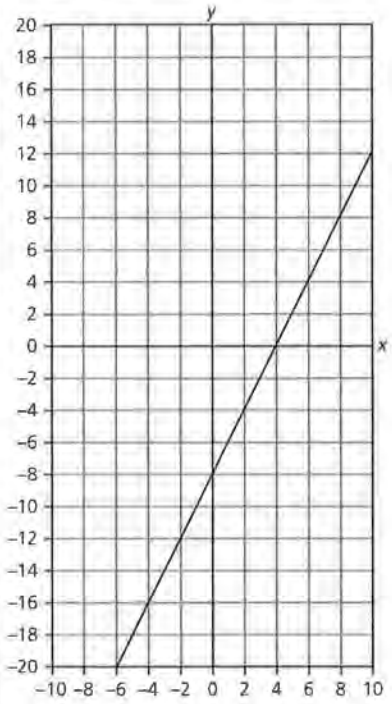
This lesson requires the use of technology. As in Lessons 4 and 5, the calculations necessary to make intelligent decisions would become so tedious that students' understanding would be impaired.

Solution Key

Discussion and Practice

1. In the line $y = 2x - 8$, the slope of the line is 2 and the y-intercept is -8.

2. The point (4, 0) is where the line crosses the x-axis. When 4 is used to replace x in the equation, it gives a y-value of zero.



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LESSON 7

Quadratic Functions and Their Graphs

How can you find the minimum point of the graph of a quadratic function algebraically?

The quadratic function created by summing the squares of the residuals was used to determine the minimum sum residuals because the graph would always have a minimum point. The goal of previous work was to find the bottom or minimum point of the curve both graphically and numerically.

OBJECTIVES

Find and interpret the x-intercepts of a quadratic equation.

Find a formula to determine the coordinates of the vertex of a parabola.

INVESTIGATE

In this lesson and the next, algebra will be used to find the minimum point. Every straight line can be described by many equivalent equations, and different forms of the equation reveal different characteristics of the line. This lesson investigates the corresponding issues for quadratic functions.

Discussion and Practice

Linear Functions

1. Consider the equation $y = 2x - 8$. What is the significance of the 2 and the -8? How do they relate to the graph?
2. Consider the equation $y = 2(x - 4)$. Graph the line. What is the significance of the point (4, 0)? How does the point relate to the equation and to the graph?

The x-coordinate of the point at which the graph of an equation crosses the x-axis is the x-intercept of the equation. This point is called a *zero* of the equation, since the y-value of the equation is zero at that point.

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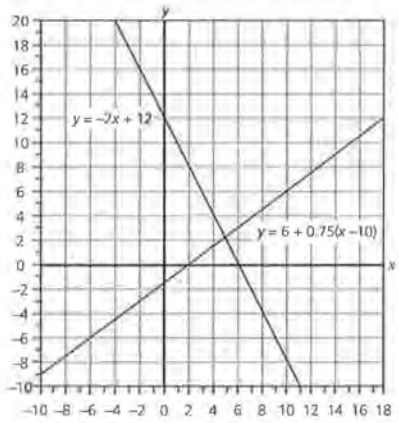
3. Study the graph and the equation for each line in the plot below.



   a. What are the slope and the x-intercept for each equation?

   b. Use the graph to explain why the x-intercept is called a *zero*.

Quadratic Functions

The zeros of a quadratic function behave the same as the zeros of a linear function. They make the y-value of the function zero.

4. The zeros of a quadratic can be found in several ways. Consider the equation $y = x^2 - 3x - 10$.

   a. Use a spreadsheet or calculator to create a table to help you find the value(s) of x that will make $y = 0$.

   b. Graph the equation. Where does the graph cross the horizontal axis?

   c. What is the solution to the equation $0 = x^2 - 3x - 10$? How does this solution relate to the graph?

5. Graph the equation $y = (x - 2)(x + 3)$.

   a. Describe the graph.

   b. What are the zeros of the equation?

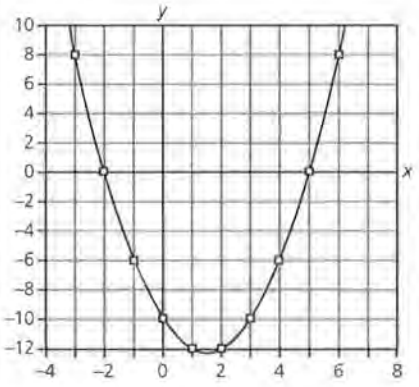
3. **a.** For the line $y = -2x + 12$, the slope is -2 and the x-intercept is 6. For the line $y = 6 + 0.75(x - 10)$, the slope is 0.75 and the x-intercept is 2.

b. The x-intercept is the point whose y-value is zero.

4. **a.** Sample table:

x	y
−3	8
−2	0
−1	−6
0	−10
1	−12
2	−12
3	−10
4	−6
5	0
6	8

b. The graph crosses the x-axis at -2 and 5.

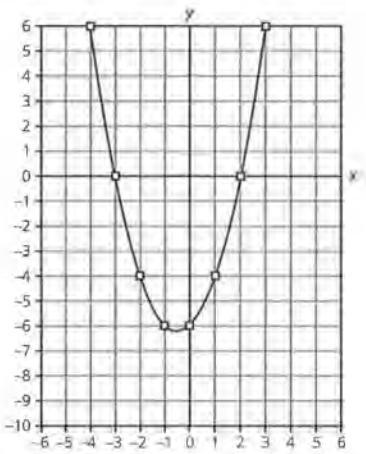


c. The solutions are $x = -2$ and $x = 5$. These are the values of the x-intercepts.

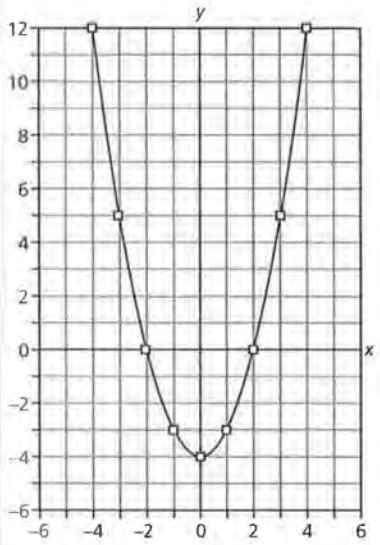
5. See graph at right.

a. The graph is a parabola with the minimum point at $x = -0.5$.

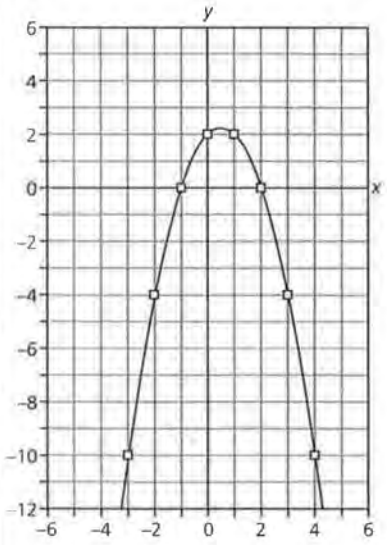
b. The zeros occur when $x = 2$ and when $x = -3$.



6. a. Crosses the x-axis at -2 and 2



b. Crosses the x-axis at -1 and 2



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6. Graph each equation. Describe the points at which each graph intersects the horizontal axis.

a. $y = x^2 - 4$ **b.** $y = -x^2 + x + 2$

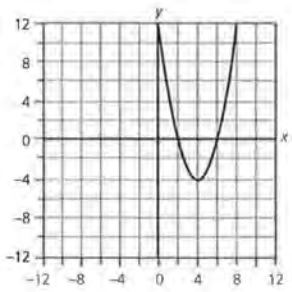
c. $y = x^2 + 2x - 24$ **d.** $y = x^2 - 3x - 5$

e. $y = 2x^2 - x - 2$ **f.** $y = -x^2 - 2$

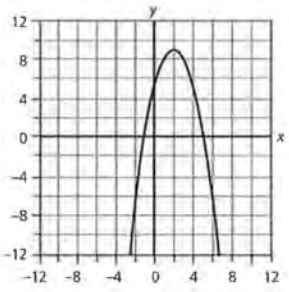
7. The graph of a function may have minimum points, maximum points, both minimum and maximum points, or neither minimum nor maximum points. Estimate what you think might be the x-coordinate of a minimum or maximum point for each graph in Problem 6. Explain why you selected those points and how those points are related to the x-intercepts.

8. Describe each of the following graphs in terms of its x-intercepts, symmetry, and minimum or maximum point.

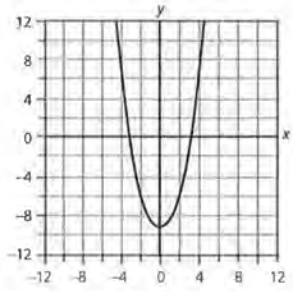
a.



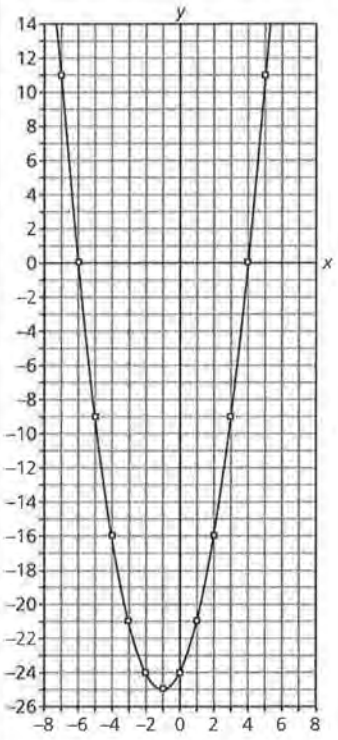
b.



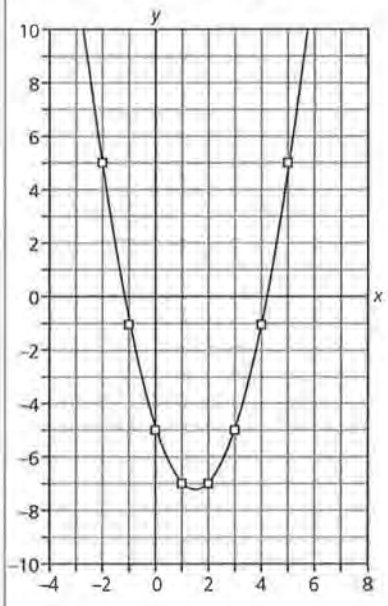
c.



c. Crosses the x-axis at -6 and 4



d. Crosses the x-axis at about -1.19 and about 4.19



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d. How does knowing the x-intercepts, or zeros, of a quadratic equation, along with an understanding of symmetry, help you to find the x-coordinate of a maximum or minimum point?

9. How can you find the minimum point of a quadratic function? Use your technique to find the minimum point for $y = x^2 - 10x + 16$.

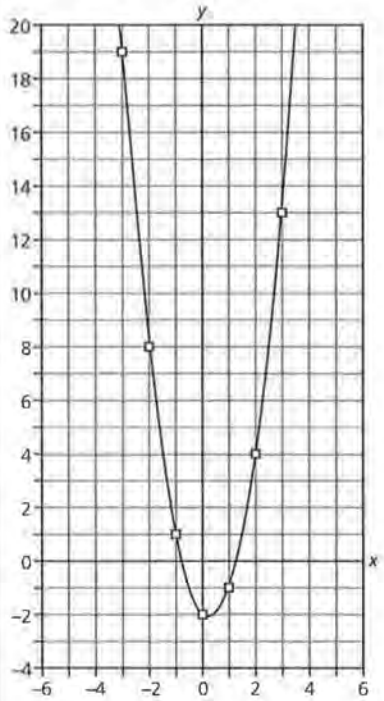
10. A parabola is the graph of a quadratic equation of the form $y = ax^2 + bx + c$ where a, b, and c represent constants in the equation and $a \neq 0$. The table below is also on *Activity Sheet 6*.

a. Give the value of a, b, and c; the x-intercepts; and the maximum or minimum point for each equation.

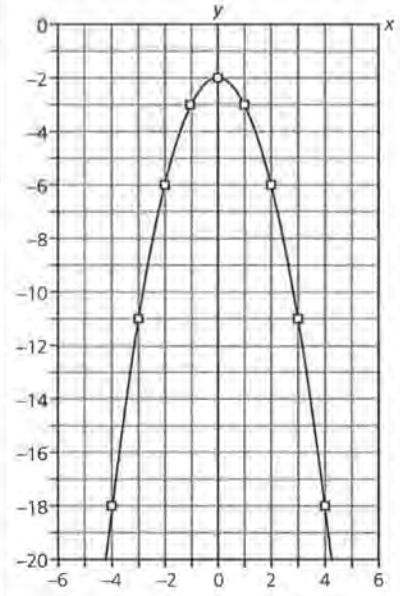
Equation	a	b	c	x-Intercepts	Minimum/ Maximum
$y = -2x^2$					
$y = 4x^2$					
$y = x^2 - 10x + 16$					
$y = x^2 - 10x - 11$					
$y = 3x^2 + 13x + 4$					
$y = x^2 - x - 2$					
$y = 8x^2 - 18x + 7$					
$y = x^2 - x + 2$					
$y = 2x^2 - x - 1$					
$y = x^2 - 2x + 1$					
$y = 3x^2 - 2x - 1$					
$y = x^2 - 4$					
$y = 9 - x^2$					

b. Find a pattern or relationship between the x-coordinate of the minimum or maximum point and the coefficients in the equation; that is, tell how the x-coordinate of the minimum or maximum point depends on a, b, and/or c in the equation. Test your conjecture with these two examples: $y = x^2 - 2x - 24$ and $y = 2x^2 - 3x - 5$.

(6) e. Crosses the x-axis at about –0.78 and about 1.28



f. Does not cross the x-axis



7. If the graph has x-intercepts, the x-coordinate of the maximum or minimum point is halfway between them.

Equation	a	b	c	x-Intercepts	Minimum/ Maximum
$y = -2x^2$	–2	0	0	(0, 0)	(0, 0)
$y = 4x^2$	4	0	0	(0, 0)	(0, 0)
$y = x^2 - 10x + 16$	1	–10	16	(2, 0), (8, 0)	(5, –9)
$y = x^2 - 10x - 11$	1	–10	–11	(–1, 0), (11, 0)	(5, –36)
$y = 3x^2 + 13x + 4$	3	13	4	$(-4, 0), (-\frac{1}{3}, 0)$	$(-\frac{13}{6}, -\frac{121}{12})$
$y = x^2 - x - 2$	1	–1	–2	(2, 0), (–1, 0)	$(\frac{1}{2}, -\frac{9}{4})$
$y = 8x^2 - 18x + 7$	8	–18	7	$(\frac{1}{2}, 0), (\frac{7}{4}, 0)$	$(\frac{9}{8}, -\frac{25}{8})$
$y = x^2 - x + 2$	1	–1	2	None	$(\frac{1}{2}, \frac{7}{4})$
$y = 2x^2 - x - 1$	2	–1	–1	$(-\frac{1}{2}, 0)$, (1,0)	$(\frac{1}{4}, -\frac{9}{8})$
$y = x^2 - 2x + 1$	1	–2	1	(1, 0)	(1, 0)
$y = 3x^2 - 2x - 1$	3	–2	–1	$(-\frac{1}{3}, 0)$, (1, 0)	$(\frac{1}{3}, -\frac{4}{3})$
$y = x^2 - 4$	1	0	–4	(–2, 0), (2, 0)	(0, –4)
$y = 9 - x^2$	–1	0	9	(–3, 0), (3, 0)	(0, 9)

a. Minimum point = (0, –4)

b. Minimum point = (0.5, –2.25)

c. Minimum point = (–1, –25)

d. Minimum point = (1.5, –7.5)

e. Minimum point = (0.25, –2.125)

f. Minimum point = (0, 2)

8. a. The x-intercepts are $x = 2$ and $x = 6$; the minimum point is (4, –4); the graph is symmetric about the line $x = 4$.

b. The x-intercepts are $x = -1$ and $x = 5$; the maximum point is (2, 9); the graph is symmetric about the line $x = 2$.

c. The x-intercepts are $x = -3$ and $x = 3$; the minimum point is (0, –9); the graph is symmetric about the line $x = 0$.

d. It appears that the minimum (or maximum) is halfway between the x-intercepts, if there are any.

9. First find the x-intercepts and then find the x-coordinate of the point halfway between. Substitute this x-value in the equation to find the value of y.

$0 = x^2 - 10x + 16$

$0 = (x - 8)(x - 2)$

x-intercepts at 2 and 8

The minimum is at $x = 5$. The minimum point is (5, –9).

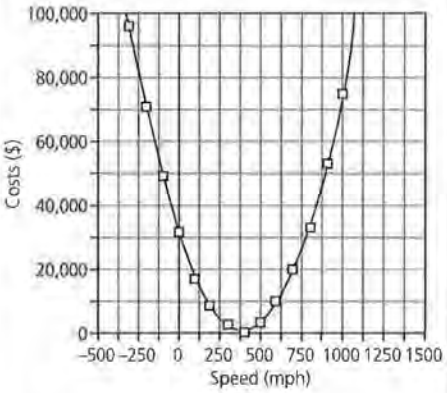
10. a. See table above.

b. $x = \frac{-b}{2a}$. For the first equation, the x-coordinate of the minimum point is 1. For the second, the x-coordinate of the minimum is $\frac{3}{4}$.

c. The formula does not work if $a = 0$, because division by zero is not defined. Of course, if $a = 0$, the function is a line and not a parabola.

d. The value of c does not play a role in the x-coordinate of the minimum or maximum point.

11. a. The minimum point is (389.75, 831), which means that if a plane goes 389.75 miles per hour, its minimum operating cost will be \$831 per hour.



b. One would suspect that the times that a plane goes slower are when it is taking off or landing. So, it is reasonable that this would use more gas and thus cost more. However, it seems unreasonable that when the plane is not moving that it would cost \$31,212, so there seems to be a problem with this equation at low speeds.

c. There are no x-intercepts; this means that it never costs nothing to operate the plane. The y-intercept is 31,212, which means that it costs \$31,212 to have the plane stopped. This may not be realistic, as mentioned in part b.

d. The formula predicts that it would cost \$17,622 per hour to operate at 100 miles per hour. This is significantly more than \$45. So, it seems the formula is not appropriate for a Piper Cub.

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c. What happens in the formula in your conjecture above if $a = 0$?

d. Does the value of c help you find the x-coordinate of the minimum or maximum point?

If the equation is written in the form $y = ax^2 + bx + c$, the x-coordinate of the minimum or maximum point can be found by using the formula $x = \frac{-b}{2a}$. Will that rule always work? Try it with the equations in the table above.

11. A researcher studied the operating cost and speed for commercial jet planes. As a result, the equation $C = 0.2S^2 - 155.9S + 31{,}212$ was produced as a model for C, the cost in dollars per hour of operating an airplane in terms of the airborne speed, S, in miles per hour.

a. Graph the equation. What is the minimum point and what does it mean?

b. Is it realistic to think that the operating costs will be greater for lower speeds?

c. Find the intercepts and determine if they make sense in terms of the situation.

d. Do you think the formula will apply to the Piper Cub, which has an operating cost of \$45 per hour and an airborne speed of 100 miles per hour? Explain why or why not.

12. Consider the equation $y = ax^2 + bx + c$.

a. Find y when $x = \frac{-b}{2a}$.

b. Explain what the value of y represents.

c. What happens when $a = 0$?

12. a. $y = \frac{4ac - b^2}{4a}$

b. It will represent the y-value of the maximum or minimum point, which is the vertex of the parabola.

c. If $a = 0$, the function is linear.

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Summary

A parabola is any equation of the form $y = ax^2 + bx + c$, $a \neq 0$. The graph of a parabola will be a U-shaped curve. The vertex of the parabola is where the minimum or maximum occurs. If the graph crosses the x-axis, the x-coordinate of the vertex can be found by taking the average of the x-intercepts. A formula can also be used to find the x-coordinate of the vertex. If $a > 0$, the curve opens up, and the minimum point is at $x = \frac{-b}{2a}$.

If $a < 0$, the curve opens down, and the maximum point is at $x = \frac{-b}{2a}$. To find the maximum or minimum y-value, evaluate the expression $y = \frac{-b^2 + 4ac}{4a}$.

LESSON 8

The Least-Squares Line

Materials: graph paper, rulers, *Activity Sheet 7, Lesson 8 Quiz*
Technology: graphing calculator or computer spreadsheet program
Pacing: 1–2 class periods and homework

Overview

The original problem proposed trying to find an equation to minimize the sum of the squared residuals. A residual is the difference between the observed value and the predicted value for a given x-value. The smaller the sum of squared residuals, the smaller the root mean squared error will be in the predictions. Finding an equation that will give this minimum requires finding the slope of that equation and a point through which the equation passes. Earlier investigations explored slopes and intercepts to find the line that gives the minimum sum of squared residuals and resulted in a parabolic function. The minimum value of the sum of squared residuals occurs at the vertex of the parabola. This lesson uses the mathematics just studied to find an equation for the least-squares line for the residuals.

Teaching Notes

This lesson requires the students to have the knowledge of summation and the summation symbol. If you have an average class, it is recommended that this lesson be used as a summary of Lessons 4, 5, and 6, to be accomplished in approximately $\frac{1}{2}$ to 1 class period.

If you have an advanced class, you will probably cover one or both of the options, in which case it is likely to take 1 to 2 class periods for students to complete the lesson. At the end of this lesson, demonstrate to the students how to use the graphing calculator to determine an equation of the regression line.

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LESSON 8

The Least-Squares Line

Can an equation be found that will minimize the sum of the squared residuals?

How do you find the least-squares line for a set of data?

OBJECTIVE

Understand the mathematics behind the least-squares line.

Return to the original problem of trying to find an equation to minimize the sum of squared residuals. Remember that a residual is the difference between the observed y-value and the predicted y-value for a given x-value. The less the sum of squared residuals, the less the root mean squared error will be in the predictions. Finding an equation that will give this minimum requires finding the slope of that equation and a point through which the graph of the equation passes.

INVESTIGATE

Earlier investigations that explored slopes and intercepts to find the line that gives the minimum sum of squared residuals resulted in a parabolic function. The minimum value of the sum of squared residuals occurs at the vertex of the parabola. In this lesson, you will use the mathematics you just studied to find an equation for the least-squares line for the residuals.

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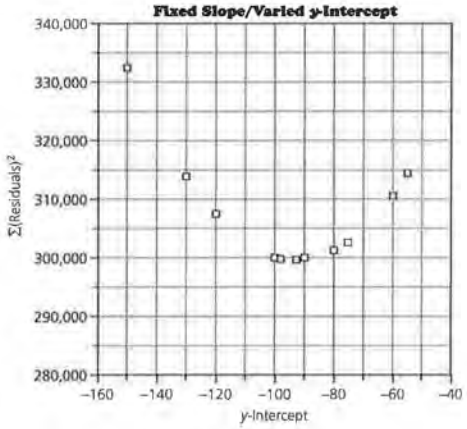
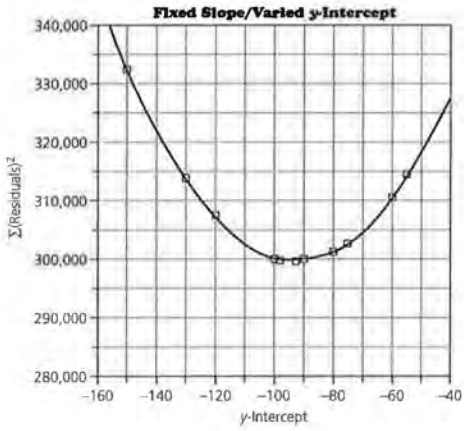
Discussion and Practice

Study the four graphs that follow.





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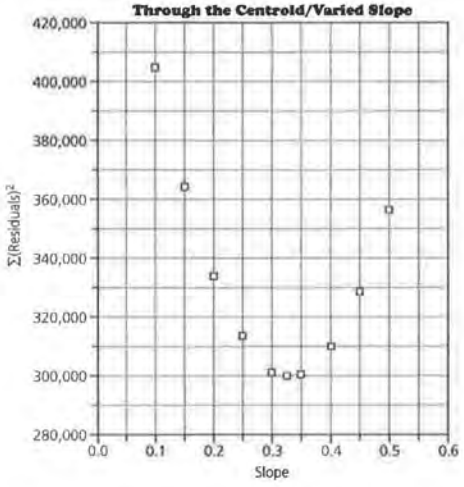

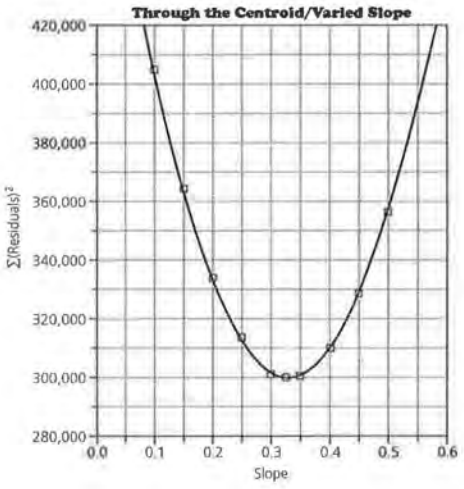

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The goal is to find the equation of a line that has the least sum of squared residuals. In order to do this, you must have a point and the slope of that line. The point that gave the minimum residual was the centroid, $(\bar{x}, \bar{y})$. Finding the slope of the equation is a challenging task. Investigating many different values, plotting the graph, and finding the coordinates of the minimum point led to the slope. A better way is to find a formula for the slope using algebra and the characteristics of a quadratic function.

Option I: Generalizing the (Number of Screens, Box-Office Revenue) Data

Remember the steps you used in earlier lessons. Calculate the difference between the actual revenue for the movies that week and the amount of money to be earned predicted by the equation of the line. Square each difference, or residual, and calculate the total sum for all the movies. Considering one movie at a time, you can find a formula. Call the slope s. Use the actual data for each movie to determine an equation with the slope s as a variable.

There were 1878 screens showing *Wayne's World,* and the actual income was 964 ten thousand dollars, or $9,640,000. Using the averages for the number of screens and the income (1418.2, 375.1) as the base point for the line gives the following equation to predict how much a movie should earn:

$y = 375.1 + s(x - 1418.2)$, where s represents slope.

Thus, to find the square of the residual for *Wayne's World,* (1878, 964), you would use the revenue of 964 ten thousand dollars minus the predicted income for the 1,878 screens on which *Wayne's World* was shown or:

$$\begin{aligned}\text{residual squared} &= (\text{observed} - \text{predicted})^2 \\ &= [y - [375.1 + s(x - 1418.2)]]^2.\end{aligned}$$

Using the information for *Wayne's World,* substitute 1878 for x and 964 for y, and the expression becomes

$[964 - [375.1 + s(1878 - 1418.2)]]^2$.

This expression can be simplified:

$$\begin{aligned}&[964 - [375.1 + s(1878 - 1418.2)]]^2 \\ &= [964 - [375.1 + 459.8s)]^2 \\ &= (964 - 375.1 - 459.8s)^2 \\ &= (588.9 - 459.8s)^2\end{aligned}$$

Solution Key

Discussion and Practice

1. s is the variable representing the possible values of the slope of the line used to make the predictions.

2. **a.** It is a parabola because its equation is a quadratic equation.

b. To find the x-coordinate of the minimum point, use $\frac{-b}{2a}$:

$\frac{541{,}552.44}{(2)(211{,}416.04)} = 1.28$. Then use this value in the function to get the minimum point of (1.28, 0.1267). This means that if a slope of 1.28 is used, the squared residual will be 0.1267.

c. 109,362.49

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Squaring the binomial yields:

$(588.9 - 459.8s)^2$
$= (588.9 - 459.8s)(588.9 - 459.8s)$
$= 588.9(588.9 - 459.8s) - 459.8s(588.9 - 459.8s)$
$= 588.9 \cdot 588.9 - 588.9 \cdot 459.8s - 459.8s \cdot 588.9 + 459.8s \cdot 459.8s$
$= 346{,}803.21 - 541{,}552.44s + 211{,}416.04s^2$

1. What does s represent?

2. Consider the graph of $y = 346{,}803.21 - 541{,}552.44s + 211{,}416.04s^2$.

a. How do you know the graph is a parabola?

b. Find the minimum point and indicate what that point represents.

c. Find the squared residual when s is 2.

3. To help you investigate the sum of squared residuals, find the corresponding expression for each of the other movies by completing a table like the following, or use *Activity Sheet 7*. Be careful with the quantities and squaring terms.

Movie	Screens	Income ($)	Predicted Income ($) $s(x - 1418.2) + 375.1$	Squared Residual	Quadratic-Error Expression
Wayne's World	1878	964	$s(1878 - 1418.2) + 375.1$	$[964 - (459.8s + 375.1)]^2$	$346{,}803.21 - 541{,}552.44s + 211{,}416.04s^2$
Memoirs of an Invisible Man	1753	460	_____	_____	_____
Stop or My Mom Will Shoot	1963	448	_____	_____	_____
Fried Green Tomatoes	1329	436	_____	_____	_____
Medicine Man	1363	353	_____	_____	_____
The Hand That Rocks the Cradle	1679	352	_____	_____	_____
Final Analysis	1383	230	_____	_____	_____
Beauty and the Beast	1346	212	_____	_____	_____
Mississippi Burning	325	150	_____	_____	_____
The Prince of Tides	1163	146	_____	_____	_____

3.

Movie	Screens	Income ($)	Predicted Income ($) $s(x - 1418.2) + 375.1$	Squared Residual	Quadratic-Error Expression
Wayne's World	1878	964	$s(1878 - 1418.2) + 375.1$	$[964 - (459.8s + 375.1)]^2$	$346{,}803.21 - 541{,}552.44s + 211{,}416.04s^2$
Memoirs of an Invisible Man	1753	460	$s(1753 - 1418.2) + 375.1$	$[460 - (334.8s + 375.1)]^2$	$7208.01 - 56{,}849.04s + 112{,}091.04s^2$
Stop or My Mom Will Shoot	1963	448	$s(1963 - 1418.2) + 375.1$	$[448 - (544.8s + 375.1)]^2$	$5314.41 - 79{,}431.84s + 296{,}807.04s^2$
Fried Green Tomatoes	1329	436	$s(1329 - 1418.2) + 375.1$	$[436 - (-89.2s + 375.1)]^2$	$3708.81 + 10{,}864.56s + 7956.64s^2$
Medicine Man	1363	353	$s(1363 - 1418.2) + 375.1$	$[353 - (-55.2s + 375.1)]^2$	$488.41 - 2439.84s + 3047.04s^2$
The Hand That Rocks the Cradle	1679	352	$s(1679 - 1418.2) + 375.1$	$[352 - (260.8s + 375.1)]^2$	$533.61 + 12{,}048.96s + 68{,}016.64s^2$
Final Analysis	1383	230	$s(1383 - 1418.2) + 375.1$	$[230 - (-35.2s + 375.1)]^2$	$21{,}054.01 - 10{,}215.04s + 1239.04s^2$
Beauty and the Beast	1346	212	$s(1346 - 1418.2) + 375.1$	$[212 - (-72.2s + 375.1)]^2$	$26{,}601.61 - 23{,}551.64s + 5212.84s^2$
Mississippi Burning	325	150	$s(325 - 1418.2) + 375.1$	$[150 - (-1093.2s + 375.1)]^2$	$50{,}670.01 - 492{,}158.64s + 1{,}195{,}086.24s^2$
The Prince of Tides	1163	146	$s(1163 - 1418.2) + 375.1$	$[146 - (-255.2s + 375.1)]^2$	$52{,}486.81 - 116{,}932.64s + 65{,}127.04s^2$

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3. **a.** $52{,}486.81 - 116{,}932.64s + 65{,}127.04s^2$

b. They are all parabolas.

4. **a.** When three quadratic expressions are added, the result is another quadratic expression, unless the three x^2 terms cancel one another out. This is because the result is the combination of like terms and there will still be a quadratic term.

b. The sum is another quadratic expression. The graph will be a parabola.

5. **a.** 514,868.9

b. $-1{,}300{,}187.6s$

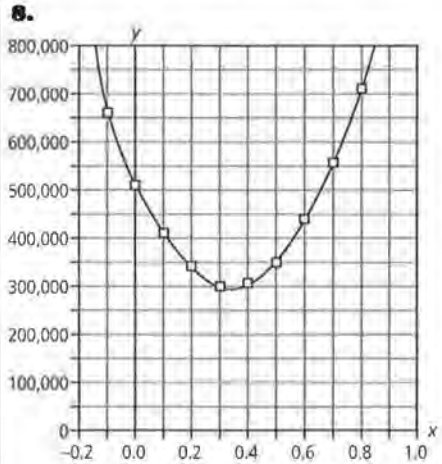
c. $1{,}965{,}999.6s^2$

6. **a.** $514{,}868.9 - 1{,}300{,}187.6s + 1{,}965{,}999.6s^2$

b. The result represents the sum of the squared residuals for a slope of 0.2.

7. They are about the same.

8.



a. $a = 1{,}965{,}999.6$; $b = -1{,}300{,}187.6$; $c = 514{,}868.9$

b. The minimum using $x = \frac{-b}{2a}$ is (0.33, 299,904.35). This means that a slope of 0.33 gives a minimum sum of squared residuals of 299,904.35.

a. What is the expression for the squared residual for *The Prince of Tides?*

b. Describe the graph of (slope, squared residual) for each movie.

Each calculation gives a formula for squared residuals for an individual movie based on slope. The result was a formula for that movie. To find the sum of the squared residuals for all of the movies, you can add the values of the individual movies.

4. Recall your earlier work combining functions in Lesson 3.

a. What kind of graph do you have when you add three quadratic expressions?

b. If you combine the individual formulas for each movie, what kind of function will you have? Describe its graph.

5. Use the expressions in the last column of the table for the movies.

a. Find the sum of all of the constant terms in that column.

b. Find the sum of all of the linear terms in that column.

c. Find the sum of all of the quadratic terms in that column.

6. Use the results of Problem 5 for the following.

a. Write an equation for the sum of squared differences between the amount of revenue given and the amount predicted for each movie.

b. Explain what your result represents when $s = 0.2$.

7. When you were estimating lines in the earlier section, you had a slope of 0.45 and found the sum of squared differences to be 327,886. How does this compare to the results you will get using your formula?

8. Graph the equation you found in Problem 6.

a. A quadratic has the form $y = ax^2 + bx + c$, $a \neq 0$. Find a, b, and c in the equation for Problem 6.

b. Use the formula from Lesson 5 to find the minimum point. What does this point represent?

9. Compare the graph of the curve generated by the equation with the graph you obtained by selecting different values for s and plotting the resulting sum of squared differences. Describe how you made your comparison.

9. They should be the same. Students may compare by matching different points in the plot and the graph.

10. a. $y = 0.33(1700 - 1418.2) + 375.1 = 468.094$; the owners would expect to make \$4,680,940.

b. Because the root mean squared error is \$1,730,000, it would be more reasonable for the owners to expect to make between \$4,680,940 – \$1,730,000 and \$4,680,940 + \$1,730,000, that is, between \$2,950,940 and \$6,410,940.

Extension

11. Answers will vary; $y = 0.29x - 93$ has 1231.94 for its sum of the absolute residuals.

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Summary

You have studied the relationship between the number of screens for a given movie and the amount of money that movie earned in a week. In attempting to find a line that seems to best summarize the relationship, you found the least-squared sum of residuals for the actual money the movie earned and the amount of money predicted by that line. If you assume that the *best* line passes through the average number of movie screens and the average amount of money earned, the equation in terms of the slopes of the line is a quadratic function. This equation has a minimum point for which a slope will give the least sum of squared differences. For this set of data, the point turns out to be (0.33, 299,894). For a slope of 0.33, the least sum of squared differences is 299,894. Thus, for a given prediction, the average root mean squared error would be \$10,000 times the square root of $\frac{299,894}{10}$, approximately 173 ten thousand dollars, or \$1,730,000, which represents the average difference between the mean and the amount of money a movie earned.

10. Suppose a movie had been shown on 1700 screens.

a. Use the line you found, the least-squares line, to predict how much income the theater owners would have expected the movie to earn that week.

b. Explain how the root mean squared error affects your prediction.

Extension

11. Find a line that seems to minimize the sum of the absolute residuals.

Option II: The General Formula

Often, a mathematical formula can be found to generalize a situation. A formula may be created in which s, the slope, is the variable. An example follows.

For each individual movie, you calculated the squared residuals, $(y_i - [375.1 + s(x_i - 1418.2)])^2$, between the actual revenue and the revenue predicted by the line. To find the sum of squared residuals, you added the squared residuals for all of the movies.

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12. a. $\Sigma(y_i - 375.1)^2$

b. $2s\Sigma(y_i - 375.1)(x_i - 1418.2)$

In symbols, this is what you calculated:

$$\Sigma[y_i - (375.1 + s(x_i - 1418.2))]^2$$
$$= \Sigma[(y_i - 375.1) - s(x_i - 1418.2)]^2$$

Simplify the expression above to

$$= \Sigma[(y_i - 375.1)^2 - 2s(y_i - 375.1)(x_i - 1418.2) + s^2(x_i - 1418.2)^2].$$

Because $\Sigma(a_i + b_i) = \Sigma a_i + \Sigma b_i$, the expression above is

$$= \Sigma(y_i - 375.1)^2 - \Sigma 2s(y_i - 375.1)(x_i - 1418.2) + \Sigma s^2(x_i - 1418.2)^2.$$

The slope s is a common factor and is not the variable for the summation, so $2s$ can be factored out of the second term of the summation expression and s^2 out of the third term. This gives

$$\Sigma(y_i - 375.1)^2 - 2s\Sigma(y_i - 375.1)(x_i - 1418.2) + s^2\Sigma(x_i - 1418.2)^2.$$

Therefore,

$$\Sigma(y_i - [375.1 + s(x_i - 1418.2)])^2 = \Sigma(y_i - 375.1)^2 - 2s\Sigma(y_i - 375.1)(x_i - 1418.2) + s^2\Sigma(x_i - 1418.2)^2.$$

12. Thus, the sum of squared residuals is a quadratic equation in which s, the slope, is the variable.

a. Identify the constant term in this expanded equation.

b. Which term is the linear term in this equation?

The minimum y-value for the quadratic can be found using the formula $y = \frac{-b}{2a}$.

$$y = \frac{-b}{2a} = -\frac{-2\Sigma(y_i - 375.1)(x_i - 1418.2)}{2\Sigma(x_i - 1418.2)^2}$$
$$= \frac{2\Sigma(y_i - 375.1)(x_i - 1418.2)}{2\Sigma(x_i - 1418.2)^2}$$
$$= \frac{\Sigma(y_i - 375.1)(x_i - 1418.2)}{\Sigma(x_i - 1418.2)^2}$$

Thus, this calculation gives the slope s that minimizes this quadratic. This is the slope of the least-squares line. The point (1418.2, 375.1) used in these calculations is the centroid $(\bar{x}, \bar{y})$ of the data.

13. a. $\frac{\Sigma(y_i - \bar{y})(x_i - \bar{x})}{\Sigma(x_i - \bar{x})^2}$

b. Using the slope from part a and the point $(\bar{x}, \bar{y})$, the result is

$$y = \frac{\Sigma(y_i - \bar{y})(x_i - \bar{x})}{\Sigma(x_i - \bar{x})^2}(x - \bar{x}) + \bar{y}.$$

Practice and Applications

14. a. $y = 0.330676x - 93.8646$

b. The slope is 0.330676, which is close to the value found.

c. $0.330676(1418.2) - 93.8646 = 375.1$

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13. The average number of screens and the average income can be expressed with the general expression using the centroid $(\bar{x}, \bar{y})$.

a. Write the general rule for finding the slope of the line that minimizes the sum of squared residuals between the observed value and the value predicted by the line.

b. Write an equation for the least-squares line.

In your search for a line that minimized the sum of squared residuals, you found the slope numerically by using a spreadsheet or calculator and algebraically using an iterative process. The section above develops an algebraic formula for the slope:

$$s = \frac{\Sigma(y_i - \bar{y})(x_i - \bar{x})}{\Sigma(x_i - \bar{x})^2}$$

Summary

To find the slope of the least-squares regression line, you could find the centroid and repeat either the numerical or the algebraic development. Fortunately, however, graphing calculators and statistical software for computers have the formula developed here programmed into their operating systems. You can access the least-squares line by selecting the **STAT** calculate menu and choosing **LinReg** in the TI calculators and by following the instructions provided when using other types of technology.

Regression is any algorithm used to predict y from a given x. In linear regression, the predicted y-values are a linear function of x. The particular technique investigated in this unit is called ***least-squares linear regression***, since it gives a linear-regression equation obtained through the least squares approach of minimizing the sum of squared residuals.

Practice and Applications

14. Enter the number of screens and box-office revenue data into your calculator.

a. Use **LinReg** to find the least-squares linear-regression line.

b. What is the slope of the LinReg line and how does it compare to the one you found earlier in this module?

c. Verify that the LinReg line contains the centroid (1418.2, 375.1).

15. They could use this line to predict what they will make for a given number of screens.

16. **a.** –0.156

b. $y = -0.156x + 13.35$

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15. Explain how the movie producers might use the least-squares line.

16. Use the BMX dirt-bike data from earlier lessons for each of the following.

a. Find the slope of the line that minimizes the sum of squared residuals.

b. Write an equation of the least-squares line.

LESSON 9

Using the Least-Squares Linear-Regression Line

Materials: graph paper, rulers, *Lesson 9 Quiz*
Technology: graphing calculator or computer spreadsheet program
Pacing: 1–2 class periods

Overview

This lesson is designed to have students practice using the least-squares linear-regression line as a tool to describe the linear relationship between two variables.

Teaching Notes

This lesson requires the use of technology. Once again, the calculations necessary to make intelligent decisions would become so tedious that students' understanding would be impaired. The lesson stresses the meaning of the least-squares regression line and its applications.

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LESSON 9

Using the Least-Squares Linear-Regression Line

When do you want to fit a line to a set of data?

What advantage is there to having a line to describe the relationship between variables?

INVESTIGATE

In this lesson, you will practice using the least-squares regression line that was developed in earlier lessons.

OBJECTIVE

Find and interpret the least-squares linear-regression line.

Discussion and Practice

The data on page 60 are aircraft-operating statistics from the Air Transport Association of America.

Solution Key

Discussion and Practice

1. **a.** $3334(1.75) = $5834.50

The cost of the fuel, the salary for the pilots and attendants, the cost of insurance, the cost of amenities (food, music, movies), the cost of the plane, airport fees, and maintenance could all be included.

b. The trip will need 1503(1.75) or 2630.25 gallons. If gas is $2.10 per gallon, it would cost $5523.53.

c. The plane goes 478 mph, so in 1.75 hours it can go 478(1.75), or 836.5, miles.

STUDENT PAGE 60

Aircraft-Operating Statistics

Aircraft	Number of Seats	Speed Airborne (mi/hr)	Flight Length (mi)*	Fuel Consumption (gal/hr)	Operating Cost ($/hr)
B747-100	405	519	3149	3529	6132
L-1011-100/200	296	498	1631	2215	3885
DC-10-10	288	484	1410	2174	4236
A300 B4	258	460	1221	1482	3526
A310-300	240	473	1512	1574	3484
B767-300	230	478	1668	1503	3334
B767-200	193	475	1736	1377	2887
B757-200	188	449	984	985	2301
B727-200	148	427	688	1249	2247
MD-80	142	416	667	882	1861
B737-300	131	413	605	732	1826
DC-9-50	122	378	685	848	1830
B727-100	115	422	626	1104	2031
B737-100/200	112	388	440	806	1772
F-100	103	360	384	631	1456
DC-9-30	102	377	421	804	1778
DC-9-10	78	376	394	764	1588

* Distance a plane can travel on a full tank of fuel Source: Air Transport Association of America

1. For a B767-300, the trip from Milwaukee, Wisconsin, to Washington, D.C., takes about 1 hour and 45 minutes.

a. How much are the operating costs to fly to Washington? What might be included in the cost per hour?

b. If fuel costs about $2.10 per gallon, how much will the fuel cost for the trip?

c. Approximately how far is it between Washington, D.C., and Milwaukee, Wisconsin? Explain how you arrived at your answer.

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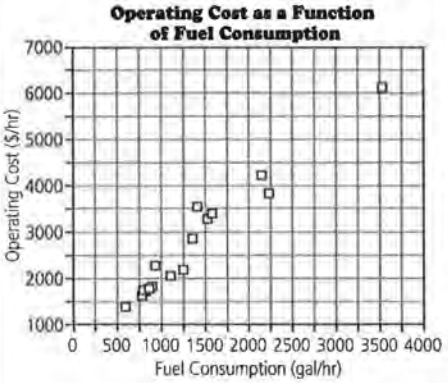
2. Plot the ordered pairs (f, c), representing the cost of operating the plane as a function of fuel used in gallons per hour.
   a. Find an equation of the least-squares linear-regression line and graph it on your plot. What is the slope and what does it tell you about the relationship between fuel and cost?
   b. Find the root mean squared error.
   c. Use your line to predict how much it would cost to operate a plane if the plane used 1000 gallons of fuel per hour. How will the root mean squared error affect your prediction?
   d. Find the number of gallons of fuel that would give a predicted cost of \$3000 to operate the plane.
   e. Verify that your least-squares line contains the centroid.
3. Plot the ordered pairs (s, c), representing the cost per hour as a function of the number of seats.
   a. Find the least-squares regression line and graph it on your plot. What is its slope and what does it tell you about the relationship between the number of seats and the operating cost per hour?
   b. How well does the line seem to describe the relationship?
   c. Calculate the cost per hour per seat for each plane. What does this tell you?

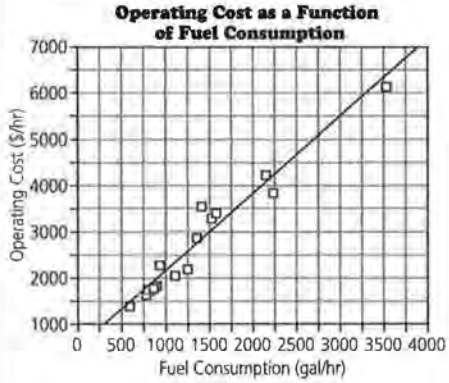
Summary

In this lesson, the least-squares regression line was used as a tool to describe the relationship between variables.

2.



a. $y = 1.646x + 522.292$; the slope is 1.646, indicating that for every increase of 1 gallon of fuel per hour, the costs increase by \$1.65 per hour.



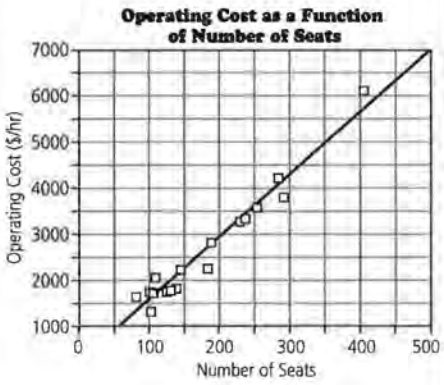
b. The root mean squared error is 246.87.

c. The equation of the line predicts a cost of \$2168.29. One would expect it would cost between \$1921.42 and \$2415.16.

d. The prediction is about 1505 gallons.

e. From the data, the centroid is (1332.88, 2716.118); $1.646(1332.88) + 522.292 = 2716.212$

3. **a.** The least-squares regression line has the equation $y = 13.712x + 174.469$. The slope is 13.172, which means that for each additional seat the operating cost increases \$13.172 per hour.



b. It seems to do quite well in most cases.

c. This is the slope of the line, \$13.17. This means that each additional seat will cost \$13.17 per hour.

STUDENT PAGE 62

Practice and Applications

The following data are from the records of the yearly passing leaders in the National Conference of the National Football League.

Passing Leaders, National Conference of NFL, 1960–1995

Passing Leaders	Attempts	Completions	Yards Earned	Touch-downs	Year
Milt Plum (CL)	250	151	2297	21	1960
Milt Plum (CL)	302	177	2416	18	1961
Bart Starr (GB)	285	178	2438	12	1962
YA Tittle (NYG)	367	221	3145	36	1963
Bart Starr (GB)	272	163	2144	15	1964
Rudy Bukich (CH)	312	176	2641	20	1965
Bart Starr (GB)	251	156	2257	14	1966
Sonny Jurgenson (WA)	508	288	3747	31	1967
Earl Morrall (BA)	317	182	2909	26	1968
Sonny Jurgenson (WA)	442	274	3102	22	1969
John Brodie (SF)	378	223	2941	24	1970
Roger Staubach (DA)	211	126	1882	15	1971
Norm Snead (NYG)	325	196	2307	17	1972
Roger Staubach (DA)	286	179	2428	23	1973
Sonny Jurgenson (WA)	167	107	1185	11	1974
Fran Tarkington (MN)	425	273	2294	25	1975
James Harris (LA)	158	91	1460	8	1976
Roger Staubach (DA)	361	210	2620	18	1977
Roger Staubach (DA)	413	231	3190	25	1978
Roger Staubach (DA)	461	267	3586	27	1979
Ron Jaworski (PH)	451	257	3529	27	1080
Joe Montana (SF)	488	311	3565	19	1981
Joe Thiesmann (WA)	252	161	2033	13	1982
Steve Bartkowski (AT)	423	274	3167	22	1983
Joe Montana (SF)	432	279	3630	28	1984
Joe Montana (SF)	494	303	3653	27	1985
Tommy Kramer (MN)	372	208	3000	24	1986
Joe Montana (SF)	398	266	3054	31	1987
Wade Wilson (MN)	332	204	2746	15	1988
Joe Montana (SF)	386	271	3521	26	1989
Phil Simms (NYG)	311	184	2284	15	1990
Steve Young (SF)	279	180	2517	17	1991
Steve Young (SF)	402	268	3465	25	1992
Steve Young (SF)	462	314	4023	29	1993
Steve Young (SF)	461	324	3969	35	1994
Bret Favre (GB)	570	359	4413	38	1995

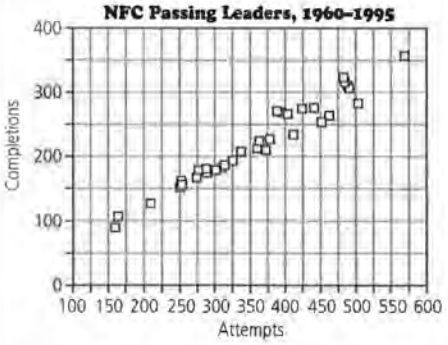
Source: *World Almanac and Book of Facts*, 1997

Practice and Applications

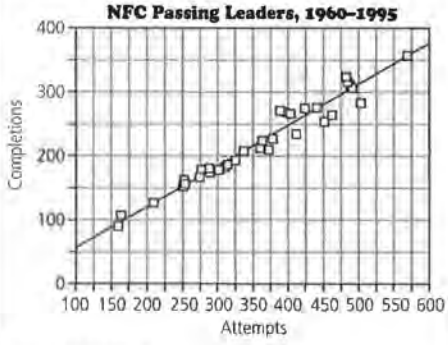
4. a. If you use attempts or completions as the number of passes, then James Harris (1976) has the greatest ratio.

b. Bret Favre in 1995 made the most touchdowns.

5. a. One would suspect that the more passes attempted, the more completed. But, those who throw only a few may complete a higher percent because they are more careful. Overall, one would expect the relationship to be basically linear.



b. The regression-line equation is $y = 0.64055x - 8.2718$. The line is the line that minimizes the squared vertical distance between the points and the line. The line passes through the centroid of the data. The y-intercept is about zero as expected. If there are no attempts, there won't be any completions. The slope of the line is about 0.6, which means that about 60% of the attempts are completions.



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4. Use a spreadsheet or your calculator for the following. Explain how you made your choice in each case.
   a. Which player has the greatest number of yards earned per pass?
   b. Which player was the best in terms of the number of touchdowns he made by passing?
5. Use the data on page 62 to answer the following.
   a. Is it true that the more passes the pass leaders attempt, the more they will complete? What do you think the relationship will be between the number of attempts and the number of completions? Graph the data for (attempts, completions).
   b. Find the least-squares linear-regression line for (attempts, completions) and write a paragraph describing how the data, the equation, and the graph are related. Include in your paragraph a description of the slope and the x- and y-intercepts. Then indicate whether either intercept makes sense in terms of the data.
   c. Investigate the relationship between the number of yards earned and the number of completions. If the relationship appears linear, find the least-squares line and explain how it applies to the data.
6. Find data on the number of passes attempted and the number of completions made by at least 10 college quarterbacks. Compare these results to those given here and use all of your information to answer this question: *Is it true that the more passes attempted the more passes completed?*

c. It appears to be linear. The linear-regression line is $y = 10.49x + 535.76$. This line has a slope of 10.49, indicating that for every completion about 10.49 yards are gained. The y-intercept is a little troubling, as it indicates that if there are no completions, about 509 yards are gained.

6. Answers will vary. The information on college quarterback ratings can be secured from the *World Sports Almanac,* or by visiting the NCAA website at www.ncaa.org.

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7. The table below shows per-capita income (in dollars) in the United States over the period 1971 through 1991. It also shows the suggested retail price for a basic Ford Mustang for those years.

Year	Selected Per-Capita Income ($)	Cost of Ford Mustang ($)
1971	4,302	3,783
1973	5,184	3,723
1975	6,053	4,906
1977	7,269	4,814
1979	9,032	5,339
1981	11,021	7,581
1983	12,216	8,466
1985	14,170	8,441
1987	15,655	9,948
1989	17,705	11,145
1991	19,133	11,873

Source: U.S. Bureau of Census, *Survey of Current Business*, April, 1992

Use these data for the following problems.

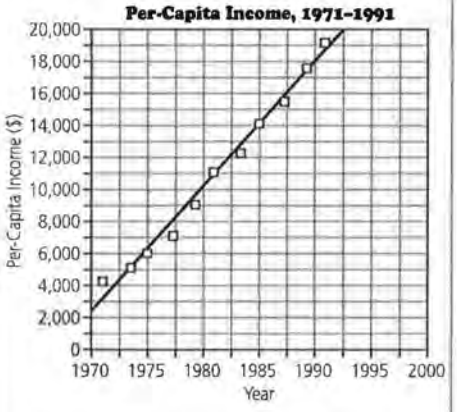
a. Plot (year, income) and observe the pattern. Fit a regression line that could be used to summarize the relationship between income and year. Interpret the slope of this line, making sure to use the proper units.

b. Plot Ford Mustang prices over the years and observe the pattern. Fit a regression line to these data. Interpret the slope of this line. How well do you think your line fits the data?

c. Calculate the percent of per-capita income required to purchase a Mustang for each year. Plot (year, percent) and observe the pattern. Fit a regression line to these data and interpret the slope.

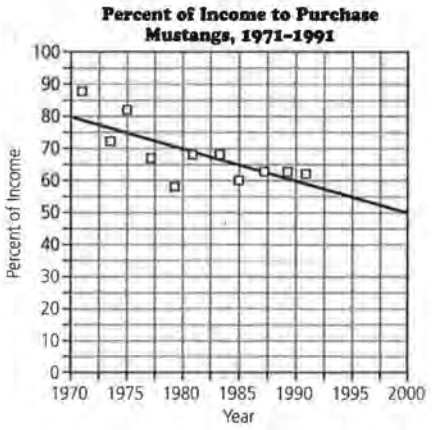
d. Considering the three plots and their regression lines, which of the three lines do you think best fits its data? Why?

7. a. The equation of the regression line is $y = 772.87x - 1{,}519{,}984.6$. The slope of 772.87 means that each year the per-capita income increased $772.87.



b. The regression-line equation is $y = 434.75x - 853{,}965.3$. The slope of this line, 434.75, means that the price of a Mustang went up about $434.75 per year. The line doesn't seem to fit as well as lines found for some other data sets.

c. There seems to be a slightly decreasing trend. The regression-line equation is $y = -1.002x + 2054.07$. The slope of about –1 means that for each year you need 1% less of the per-capita income to buy a Mustang.



d. It appears that the line for the year-versus-per-capita-income plot fits the best. In that one, the line seems to go through or very close to most of the points.

LESSON 10

Correlation

Materials: graph paper, rulers, *Activity Sheet 8, Lesson 10 Quiz*
Technology: graphing calculator or computer spreadsheet program
Pacing: 3 class periods and homework

Overview

In the previous lessons, students learned how to find the *best* linear relationship between two variables. Now the question is: Is there any general way to measure the *strength* of the relationship? For example, how closely is the amount of sugar in a cup of cereal related to the number of calories? There exists a numerical value, called the ***correlation coefficient,*** created to do just that, measure the strength of the linear relationship between two variables. This lesson investigates how the correlation coefficient is defined, what it tells you about the relationship between two variables, and what it does not tell you about that relationship. In addition, the correlation coefficient is developed from the definition of r^2. The definition of r^2 is developed both graphically and algebraically.

Teaching Notes

This lesson will require you to do some modeling and lecturing from time to time. Care must be taken, however, to not overdo the lecturing and modeling, in order that the students can more fully appreciate that they are able to—and we expect them to—read and do the mathematics.

In some instances, this lesson requires the use of technology because the calculations necessary to make intelligent decisions would become so tedious that students' understanding would be impaired. In other cases, it might be best to use paper and pencil while reading for understanding.

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LESSON 10

Correlation

Is there any general way to measure the *strength* of a linear relationship between two variables?

What is the correlation coefficient?

How well can you predict *y* when you *use* the *x*-variable?

How well can you predict *y* when you *do not use* the *x*-variable?

INVESTIGATE

In the previous lessons, you learned how to find the *best* linear relationship between two variables. Is there any general way to measure the *strength* of the relationship? For example, how closely is the amount of sugar in a cup of cereal related to the number of calories? There exists a numerical value created to do just that, that is, to measure the strength of the linear relationship between two variables. This number is called the ***correlation coefficient***. This lesson investigates how the correlation coefficient is defined, what it tells you about the relationship between two variables, and what it does not tell you. Consider the data on page 66, taken from *Consumer Reports*, November, 1992, on breakfast cereals. The data are given for a serving size of one cup.

OBJECTIVE

Find and interpret the correlation coefficient.

Solution Key

Discussion and Practice

1. **a.** They have 150 calories, 330 mg of sodium, and 13 g of sugar per serving. They are 35% sugar.

b. The percent gives the percent of the weight of the cereal that is sugar; the grams give the amount of sugar in a serving.

c. The average number of calories appears to be about 160; the average number of grams of sugar appears to be about 11.

d. Shredded Wheat may be the most healthful, as it has the least sugar and very little sodium. Oatios with Extra Oat Bran is also healthful, with no sodium and very little sugar.

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Ready-to-Eat Cereal	Calories	Sodium (mg)	Sugar (g)	Sugar (percent)
Shredded Wheat Spoon Size	140	5	0	0
Common Sense Oat Bran	130	330	8	21
Frosted Mini-Wheats	130	0	8	21
Grape-Nut Flakes	110	160	6	18
Whole Grain Wheat Chex	150	350	5	11
Whole Grain Wheaties	100	200	3	11
Total Raisin Bran	140	190	14	33
Raisin Nut Bran	220	280	16	28
Raisin Squares	180	0	12	21
Oatios with Extra Oat Bran	110	0	2	6
Nutri-Grain Almond Raisin	210	330	11	18
Crispy Wheats 'N' Raisins	130	180	13	35
Life	150	230	9	21
Multi-Grain Cheerios	100	220	6	21
Oatmeal Squares	220	270	12	21
Mueslix Crispy Blend	240	220	19	31
Cheerios	90	230	1	4
Cinnamon Oatmeal Squares	220	250	14	25
Clusters	220	280	25	14
100% Natural Whole Grain with Raisins (Low Fat)	220	30	14	24
Honey Bunches of Oats with Almonds	180	240	9	21
Low-Fat Granola with Raisins	180	90	14	29
Basic 4	170	310	11	12
Just Right with Fruit & Nuts	190	250	12	24
Apple Cinnamon Cheerios	150	240	13	35
Honey Nut Cheerios	150	330	13	35
Oatmeal Raisin Crisp	260	340	20	29
Nut & Honey Crunch	170	300	12	28

Source: *Consumer Reports*, November, 1992
(Note: The column labeled "Sugar (percent)" is computed based on the weight in grams of one cup of the specific cereal.)

Discussion and Practice

1. Use the data in the table above for the following.

a. Describe Honey Nut Cheerios in terms of the information in the table.

b. What is the difference in the number of grams of sugar and the percent of sugar for a cereal?

c. Estimate the average number of calories in a serving.

d. Which cereal seems to be the most healthful in terms of the information you have? Explain your choice.

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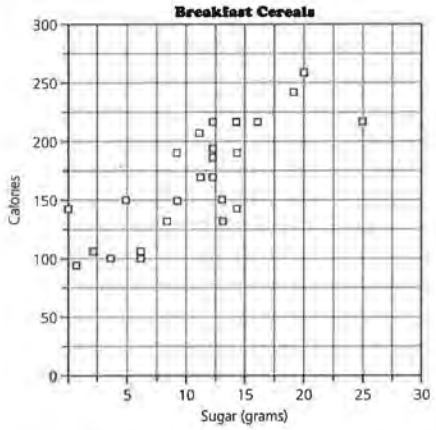
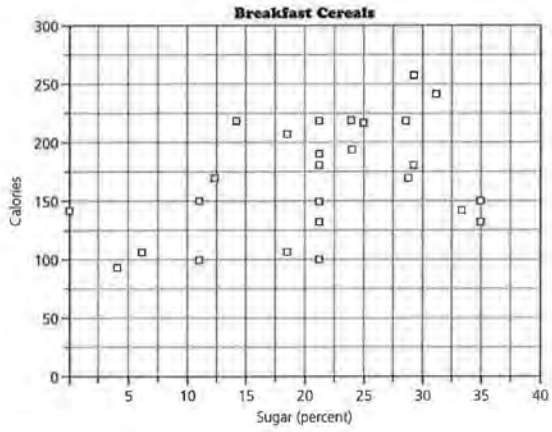
2. The plots below represent different relationships between the amount of sugar and the calories in the cereals.





a. What are the differences in the plots?

b. For which plot does the linear association seem to be stronger? How did you make your decision?

2. a. The first graph shows the amount of sugar, and the second shows the percent of sugar.

b. The connection between the grams of sugar and the calories seems to be the stronger connection, since the plot looks more linear.

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The Correlation Coefficient

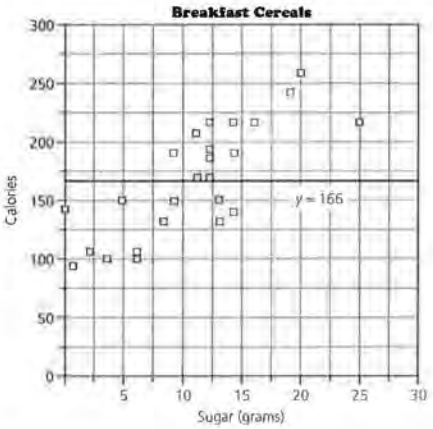
A scatter plot gives a good overall impression about the relationship between two variables. But, as the two previous plots show, it is difficult to decide exactly how strong a relationship might be or to precisely compare the strength of association in two different plots. A numerical measure of association would help. The least-squares line summarizes the linear relationship between two variables and can also be used to develop a numerical measure of association. To do so, consider two questions:

1. How well can you predict y when you *use* the x-variable?
2. How well can you predict y when you *do not use* the x-variable? For example, would it be just as accurate to estimate the y-variable based on $\bar{y}$?

Will using x make a difference? If using x does not improve the prediction of y very much, it makes sense to say that x and y are not associated very strongly. If using x greatly improves the prediction of y, then it makes sense to say that x and y are strongly associated. A statistic called the ***correlation coefficient***, which you may have encountered in different forms in earlier work, is used to turn this idea into a specific numerical measure of association.

Suppose you want to predict the number of calories in a new breakfast cereal, based on the preceding data. Using nothing other than the number of calories about the cereal—that is, not using the amounts of sodium or sugar—the best prediction is the mean number of calories from the cereals in the data, about 166.429 calories. That is, mean $y = \bar{y} = 166$ calories.

STUDENT PAGE 69



How accurate would you expect the prediction to be? What was the actual data point? The observed number of calories for Shredded Wheat Spoon Size is 140 calories, and the residual using the mean to predict would be (140 – 166). Nut & Honey Crunch has 170 calories, so the residual using the mean to predict is (170 – 166). But a measure of total error should involve all the cereals in the data, not just these two. This problem is similar to an earlier problem in this module, when you needed to find an overall measure of error from a line. There you eventually settled on the sum of squared residuals as a reasonable overall measure of error.

It makes sense to use a similar measure of error here. The only difference is that now the prediction for each cereal is mean $y = 166$. Thus, for Shredded Wheat Spoon Size, square the residual, giving $(140 - 166)^2$. For Nut & Honey Crunch, the squared residual is $(170 - 166)^2$. The total squared error using the mean calories for each prediction is found by summing these squares over all cereals in the data.

$$\Sigma(\text{observed } y_i - \text{mean } y)^2 = \Sigma(y_i - 166)^2 = 59{,}048$$

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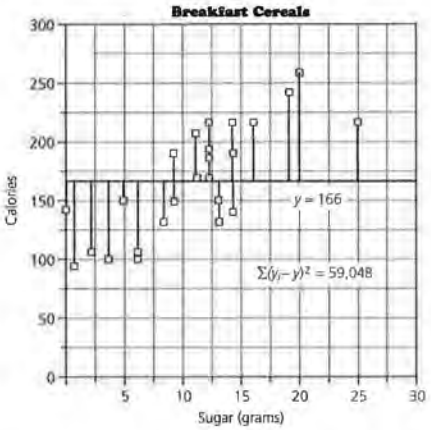

Next, suppose you use grams of sugar (the x-variable) to predict the calories (the y-variable). This is the sort of problem you have investigated throughout this module. You now know that the least-squares line gives the least sum of squared residuals. Furthermore, the total error is the sum of squared residuals from using the least-squares line for prediction, which is

Σ(observed y – predicted y using the least-squares line)^2.

In this example, for grams of sugar (x) and calories (y) the equation of the least-squares line is

$y = 6.3915x + 97.4916$, and
Σ(observed calories – predicted calories using the least-squares line from sugar)^2 = 22,493.

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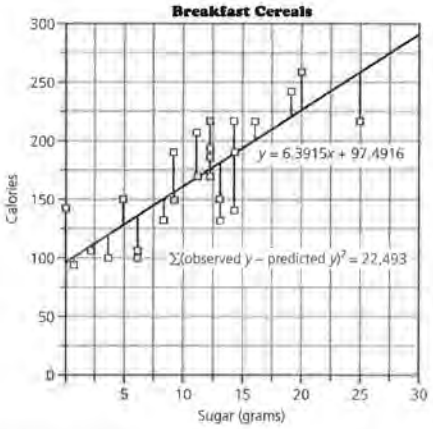

To summarize, a measure of how strongly x is associated with y can be investigated by working with the two measures of total error first derived. To predict y without using x, the square of the error is

(observed y – mean y)^2 = 59,048, or $(y - \bar{y})^2 = 59{,}048$.

To predict y by using x and the least-squares line, the square of the error is

(observed y – predicted y from least-squares line)^2 = 23,493, or $(y - \hat{y})^2 = 23{,}493$.

There is one more important fact to note about these two expressions for total error. You know that the least-squares line gives the least sum of squared residuals for any possible line. But prediction using mean $y = 166$ is equivalent to using the horizontal line with y-intercept approximately 166 and slope zero, a possible line. Thus, it will always be the case that

Σ(observed y – predicted y using least squares line)^2 $\leq \Sigma$(observed y – mean y)^2, or $\Sigma(y - \hat{y})^2 \leq \Sigma(y - \bar{y})^2$.

That is, (total error in prediction *using* x) $\leq$ (total error in prediction *not using* x).

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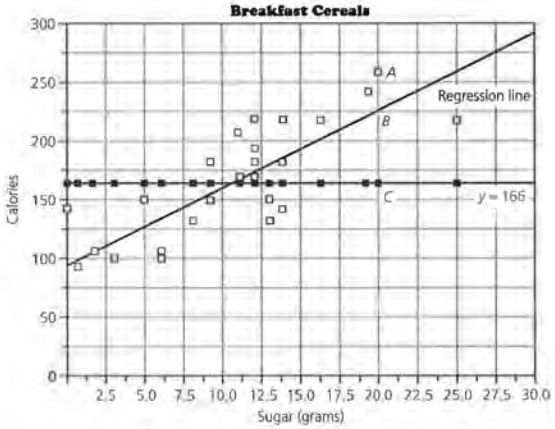
Finally, these pieces can be put together to give a measure of the strength of association between x and y. The reduction in total error is

(total error in prediction not using x – the total error in prediction using x) or, symbolically, $\Sigma(y-\bar{y})^2 - \Sigma(y-\hat{y})^2$.



Segment AC is the unexplained error strictly using $\bar{y}$ as the predictor of y. Segment BC is the part of the error accounted for, or explained, by using the regression line as the predictor of y and segment AB, the residual, is the error not explained by using the line to predict y. Thus, $AC - AB$ is the error explained by using the line, and $\frac{AC-AB}{AC}$ equals the proportion, or percent, of error explained by using the line to predict y. In other words, the proportion of reduction in total error when using x compared to not using x is

$$\frac{\left(\begin{array}{c}\text{total error in prediction}\\ \text{not using } x\end{array}\right) - \left(\begin{array}{c}\text{total error in prediction}\\ \text{using } x\end{array}\right)}{\text{total error in prediction not using } x}$$

$$= \frac{\Sigma(\text{observed } y - \text{mean } y)^2 - \Sigma(\text{observed } y - \text{predicted } y \text{ using least-squares line})^2}{\Sigma(\text{observed } y - \text{mean } y)^2}$$

$$= \frac{\Sigma(y-\bar{y})^2 - \Sigma(y-\hat{y})^2}{\Sigma(y-\bar{y})^2} = \frac{59{,}048 - 22{,}493}{59{,}048}$$

$$= 0.619$$

$$= r^2.$$

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This value is the coefficient of determination, denoted by r^2. The number r^2 represents the percent of the sum of the squared residuals that can be attributed to a linear relationship.

In the cereal data example, $r^2 = 0.62$ for calories and grams of sugar. This means that 62% of the variation in the amount of calories in a cup of cereal can be attributed to a linear relationship between calories and the grams of sugar in a cup of cereal. If you use the grams of sugar, you can predict the calories for the cereal more accurately than if you do not know anything about the sugar. In fact, in the precise mathematical sense described, you can do 62% better.

The square root of r^2, $\pm r$, is called the ***correlation coefficient***. The correlation coefficient r is a number between 1 and –1 and is a measure of linear association or the way the data points cluster around the least-squares regression line. The square of the correlation coefficient, r^2 expresses the proportion of variability in the y-variable that is explained by a change in the x-variable in the least-squares regression line. In the example above, the correlation coefficient $r = 0.78$. The formulas for finding r and r^2 have been programmed into your calculator, and your calculator can be used to find r quickly and easily for any pair of variables you have entered.

Range of r^2 and r

Notice that if all data points fall on a line, the x-variable predicts the y-variable perfectly, and each residual is zero. Thus,

$\Sigma(\text{observed } y - \text{predicted } y \text{ using least-squares line})^2 = \Sigma(y - \hat{y})^2 = 0.$

So in this case,

$$r^2 = \frac{\Sigma(\text{observed} - \text{mean})^2 - \Sigma(\text{observed} - \text{predicted})^2}{\Sigma(\text{observed} - \text{mean})^2}$$

$$= \frac{\Sigma(y - \bar{y})^2 - \Sigma(y - \hat{y})^2}{\Sigma(y - \bar{y})^2}$$

$$= \frac{\Sigma(\text{observed} - \text{mean})^2 - 0}{\Sigma(\text{observed} - \text{mean})^2} = \frac{\Sigma(y - \bar{y})^2 - 0}{\Sigma(y - \bar{y})^2} = 1.$$

And this is the greatest value for r^2, since the numerator is less than or equal to the denominator. So r^2 is always less than or equal to 1.

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Now consider the least value that r^2 could have. Refer to the formula for r^2. It shows that

Σ(observed y – mean y)^2 – Σ(observed y – predicted y using least-squares line)$^2 \geq 0$ or, symbolically, $\Sigma(y - \bar{y})^2 - \Sigma(y - \hat{y})^2 \geq 0$.

The value of r^2 is zero only when the least-squares line for the data is horizontal. This says that the slope of the least-squares line is zero, and the best prediction of y amounts to simply using the mean y regardless of the value of x.

The expression above is the numerator of r^2, and it is positive; the denominator of r^2 is also positive. So, $r^2 \geq 0$, and $0 \leq r^2 \leq 1$.

Summary

To summarize, if the data points all fall on a line, then $r^2 = 1$. If the slope of the line is positive, $r = 1$; if the slope of the line is negative, $r = -1$. One way to think of r^2 is in terms of a sliding scale from 0 to 1, where 1 is the case in which 100% of the change in y is determined by a change in x and 0 is the case in which 0% of the change in y can be explained by a change in x. Numbers between 1 and 0 indicate some amount of correlation that can be captured using words such as *strong* and *weak;* but determining exactly what amount of correlation is *strong* is quite subjective. For the calories and grams of sugar in the cereal example, an r^2 of 0.61—or correlation, r, of ± 0.78 (Use the positive value since the data show an increasing relationship.)—would usually be considered strong.

To interpret r^2 and r, it also helps to keep in mind the range of possible values and the types of scatter plots and the extreme situations to which they correspond. The value of r^2 must always satisfy $0 \leq r^2 \leq 1$. It follows that $-1 \leq r \leq 1$, since the square root can be either positive or negative. Determine which sign to use by the dependence shown in the data. The value of r^2 is 1 only when all the data points fall on a line. The value of r^2 is zero only when the least-squares line for the data is horizontal. In a case like this, the slope of the least-squares line is zero, and the best prediction of y amounts to simply using the mean of the y-values regardless of the value of x.

Look at the two plots below to get a better feeling for what r and r^2 indicate about the relationship of the data points to the least-squares regression line. For r close to 1, the points form a narrow elliptical cloud close to the line. For r close to zero, the points form a wider, or "fatter"-appearing, cloud.

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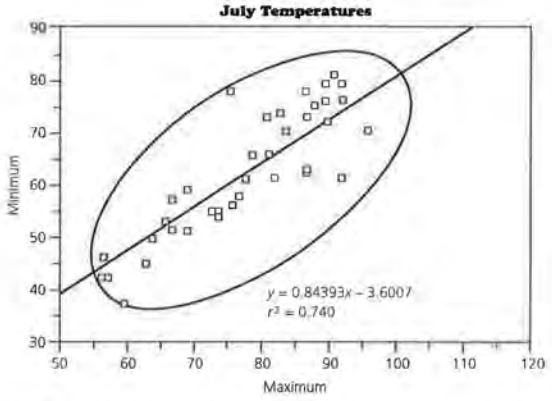
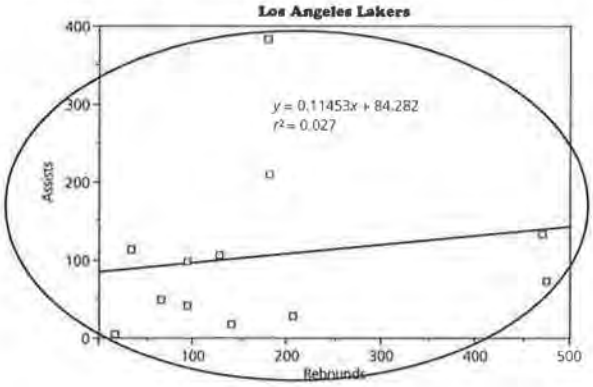


In the graph above, note that the points cluster around the line; $r^2 = 0.740$ and $r = 0.86$. There is a high degree of association between the maximum and minimum July temperatures for cities. Knowing the maximum July temperature does help predict the minimum July temperature.



Note that the points do not cluster close to the line; $r^2 = 0.027$ and $r = 0.16$. There is little association between rebounds and assists for the Lakers basketball team. Knowing the number of rebounds for a player does not help to predict the number of assists for that player.

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Units of r^2

The units that go along with the r^2 formula involve two different summation expressions. For Σ(observed y – mean y)^2, each difference has a unit of calories, so this entire summation has a unit of calories squared. Similarly, for Σ(observed y – predicted y using least-squares line)^2, each difference has a unit of calories, so the entire summation has a unit of calories squared.

Putting these together and showing the units in the formula for r^2 gives the following expression:

$$\frac{\Sigma(\text{observed} - \text{mean})^2\text{calories}^2 - \Sigma(\text{observed} - \text{predicted})^2\text{calories}^2}{\Sigma(\text{observed} - \text{mean})^2\text{calories}^2}$$

Both numerator and denominator have units calories squared. In the ratio the units reduce, and the ratio does not have any units. Thus, the number r^2 is without a unit.

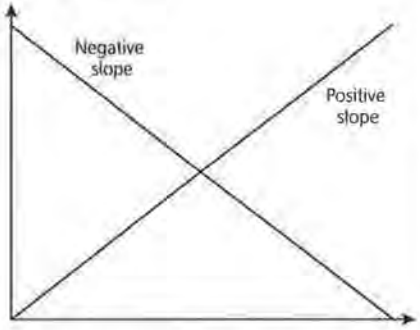
r^2 Measures the Strength of a *Linear* Relationship

You have learned that the least-squares line is the line that minimizes the sum of squared residuals among all possible lines. The number r^2 is defined for this line. Because the r^2 formula uses the sum of squared residuals from this line, r^2 is a measure of how strongly the data points follow a *linear* relationship.

Consider the relationship between grade-point average and the number of hours students study. Grade-point averages may vary from 0.0 to 4.0 on a four-point scale. Suppose the correlation, r, is around 0.7. This means that the correlation is fairly strong, and the points lie in a cloud close to the least-squares line. Calculating r^2 gives $0.7^2 = 0.49$, so 49% of the variability in grade-point averages can be explained by how much students study and the least-squares line. The remaining 51%, however, is due to other factors, such as difficulty of classes, amount and quality of homework, and differences among individual students.

The relationship between grade-point averages and the number of hours you sleep is a different story. Suppose the correlation is 0.2. There is little association; the points lie in a wide ellipse, and the least-squares line does not tightly summarize the data. Then $r^2 = 0.04$, which indicates that only 4% of the variability in grades would be attributed to the number of hours students sleep. More than 95% of the variation in grade-point averages is due to other factors.

3. a. Samples:



b. r could be 0.9 or –0.9.

4. a. The formula for r^2 is a fraction of the form $\frac{A-B}{A}$, where A is the sum of the squared differences between the observed values and the mean and B is the sum of the squared differences between the observed values and the predicted values. This fraction will always be less than or equal to 1, since B is never negative. The greatest value for this fraction occurs when $B = 0$, which gives the value of 1 for r^2. So r^2 can never be greater than 1. Since r^2 cannot be greater than 1, r must be less than or equal to 1.

b. No; if r were less than –1, then r^2 would be greater than 1, which is not possible.

c. Yes; in order for r^2 to be 1, the sum of the squared differences between the predicted and the actual values must be zero, which means that the points must all fall on the line.

5. a. The two summations must be the same.

b. The least-squares line has a slope of zero. It uses the mean of the y-values to make the prediction.

c. r^2 will be zero.

d. The variables are not strongly associated. r^2 would be zero.

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Because r is calculated using the mean y-value and the residuals from the least-squares regression line, the numerical value of r can be misleading. It is important to remember that r measures the strength of a linear relationship. It does not measure the existence of any other pattern in the data, and an r near zero does not mean another pattern might not exist. The value r is very sensitive to outliers because of the way it is calculated. Outliers can make the correlation seem strong when, in fact, little exists. Likewise, outliers can make the correlation seem weak, when, in fact, the relationship is very linear. To be sure you understand what the correlation coefficient indicates about the relationship, look at the plot of the data and check for patterns and outliers. This will help to ensure that the correlation you find makes sense in terms of the data.

3. A positive value of r corresponds to a line with positive slope. A negative value of r corresponds to a line with negative slope.

- **a.** Sketch a line to illustrate each case.
- **b.** If r^2 is 0.81, what are possible values for r?

4. Consider the range of values for r and r^2.

- **a.** Is it possible for r^2 to be greater than 1? Can r ever be greater than 1? Explain.
- **b.** Can r ever be less than –1? Explain.
- **c.** If $r^2 = 1$, must all the data points fall on a line? Explain.

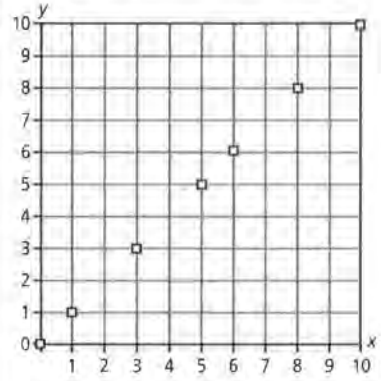
5. Suppose $r^2 = 0$.

- **a.** What can you conclude about the two summations in the numerator of the definition of r^2?
- **b.** What can you conclude about the least-squares line for these data?
- **c.** If the slope of the least-squares line is zero, what can you conclude about the value of r^2?
- **d.** Suppose the (x, y) data points fall on two parallel lines symmetrically distributed. Would you say that these x- and y-variables are strongly associated? What is r^2?

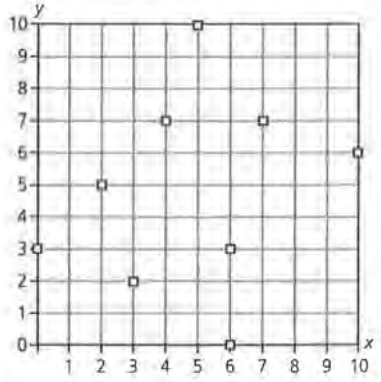
6. Draw an example of a scatter plot for each situation.

- **a.** The correlation is close to 1.
- **b.** The correlation is close to zero.

6. a. Sample:



b. Sample:



7. Answers will vary.

a. One would expect limited correlation.

b. One would expect a slightly strong correlation.

c. One would not expect any correlation.

8. **a.** $r = -0.03037$, which indicates that there is very little linear association. These points would not be tightly packed around the least-squares regression line.

b. $r = 0.99424$, which indicates that there is a strong linear association. These points would be very tightly packed around the least-squares regression line.

c. $r = 0$, which indicates no linear association. There is little reason to consider the least-squares regression line.

d. $r = 0.99208$, which indicates that there is a strong linear association. These points would be very tightly packed around the least-squares regression line.

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7. Describe the correlation you would expect to get from looking at a plot of each of the following.

a. The amount of rain and the percent of sun for a set of cities.

b. The amount of rain and percent of cloudy days for a set of cities.

c. The amount of rain and the temperature for a set of cities.

8. For each of the following data sets, determine the value of r and discuss how it relates to the data.

a. {(2, 1), (2.3, 1.5), (2.5, 2), (0.3, 1.4), (2.6, 2.3), (1.8, 1), (1.9, 0.2), (0.7, 0.8), (1, 2.6), (0.2, 2.1)}

b. {(1, 2), (1.5, 2.3), (2, 2.5), (1.4, 0.3), (2.3, 2.6), (1, 1.8), (0.2, 1.9), (0.8, 0.7), (2.6, 1), (2.1, 0.2), (30, 45)}

c. {(−2, 4), (−1, 1), (0, 0), (1, 1), (2, 4)}

d. {(−3, −14), (−1, −11), (1, −2), (3, 4), (7, 16), (10, 32)}

There are several different kinds of correlation and different procedures for finding correlation between variables, depending on the kind of data with which you are working. The correlation coefficient described above is called ***Pearson's r.*** It is widely used and concentrates on the degree of linearity in the relationship.

Summary

It would be useful to have some measure of association that

1. is free of units (such as grams and calories);
2. does not depend on the scale of measurement (such as grams or milligrams); and
3. would always help you judge the strength of the association between two variables.

Unfortunately there is no such measure.

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The correlation coefficient is a measure of association that meets the first two of these criteria but not the third. It measures the strength of the ***linear relationship*** between two variables in conjunction with the least-squares line. The linear association between two variables is measured by a number r called the ***correlation coefficient.*** For perfect positive correlation, $r = 1$; and for perfect negative correlation, $r = -1$. A positive correlation indicates that as one variable increases, the other tends to increase also; while a negative correlation indicates that as one increases, the other tends to decrease. If r is close to zero, then it is usually difficult to determine as one variable increases whether the second variable either increases or decreases.

The number r^2 indicates the proportion of variation in y that can be explained by using the least-squares regression line. The closer r^2 is to 1, the more valuable x is for predicting y. There are some important facts to remember about correlation.

- The correlation coefficient measures linear association only, rather than association in general. There may be a clear pattern in a set of data, but if it is not linear, the correlation may be close to zero. An example is the graph of a parabola.
- Correlation is a number without any units attached. Therefore, it does not depend on the units chosen for either variable.
- Many software packages calculate r automatically when they find the coefficients of the regression line. It is important, however, to look at the plot to determine whether the underlying relation is actually linear.

Practice and Applications

9. **a.** iv
b. ii
c. i
d. iii

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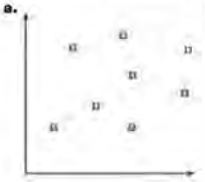
Practice and Applications

9. Match each correlation r and r^2 with the appropriate graph.

a.



b.

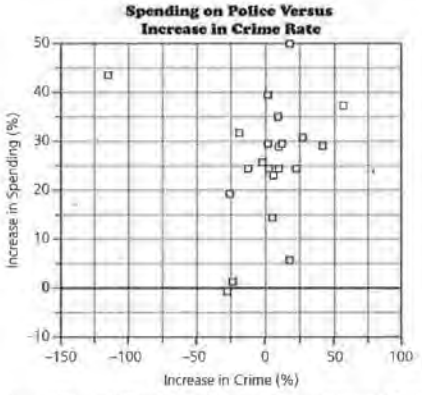
c.

d.

i. $r = -0.8$, $r^2 = 0.64$ **ii.** $r = 0.8$, $r^2 = 0.64$

iii. $r = 0.99$, $r^2 = 0.9801$ **iv.** $r = 0.15$, $r^2 = 0.0225$

10. The following plot shows how, for a set of suburban communities, spending on police increased from 1988 to 1992 and how the crime rate changed during that same time.



Source: Wisconsin Office of Justice Assistance, Wisconsin Taxpayers Alliance

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a. Give the coordinates of a point where both the increase in crime and the increase in spending are high.

b. Give the coordinates of a point where the increase in crime is low and the increase in spending is high.

c. r^2 for (% increase in crime, % increase in spending) is 0.003. What does this mean? What is r?

d. If you knew the increase in crime for a given community was 20% between 1988 and 1992, how well do you think you could predict the change in spending based on the plot?

11. Comment on this statement: *If there were no correlation, the best way to predict* y *from an* x *is just to use the mean or average* y *without any regard for the* x *with which it might be associated.*

12. Refer to the plot of the calories and grams of sugar in the beginning of the lesson.

a. Enter the data (grams of sugar, calories) into your calculator. Find the least-squares regression line and the correlation coefficient.

b. How well do you think knowing something about the number of grams of sugar will help you predict the number of calories? What did you consider in arriving at your answer?

13. Reverse your axes for (calories, grams of sugar). Make the plot.

a. Calculate the correlation coefficient. How does it compare to the correlation coefficient for (grams of sugar, calories)?

b. What is the least-squares line? How does it compare to the line for (grams of sugar, calories)? Try to explain any observations.

10. a. One possibility is (55, 38).

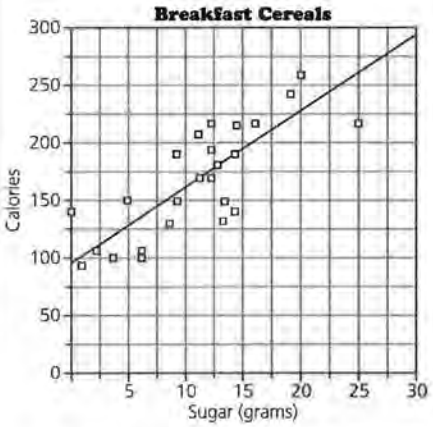
b. One possibility is (–115, 43).

c. It means that there is very little linear connection between the increase in crime and the increase in spending; $r = \pm 0.05477$

d. Only within about 0.3%

11. This statement is false. There might be a nonlinear equation that predicts *y*-values accurately.

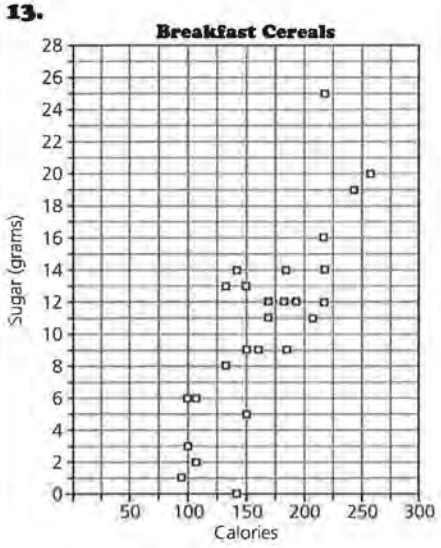
12.



a. Let C be the number of calories and S the grams of sugar. The equation is $C = 6.49S + 97.49$; $r = 0.7868$.

b. Knowing the grams of sugar will be fairly helpful in predicting the number of calories in a serving of cereal. The line in part a can be used. The value of r^2 of 0.62 means that at least 60% of the variability in the calories can be attributed to the least-squares regression line.

13.



a. $r = 0.7868$, which is the same as the correlation coefficient for (grams of sugar, calories).

b. The equation of the least-squares line is $S = 0.0969C - 5.334$. The two lines are roughly symmetric with respect to the line $y = x$.

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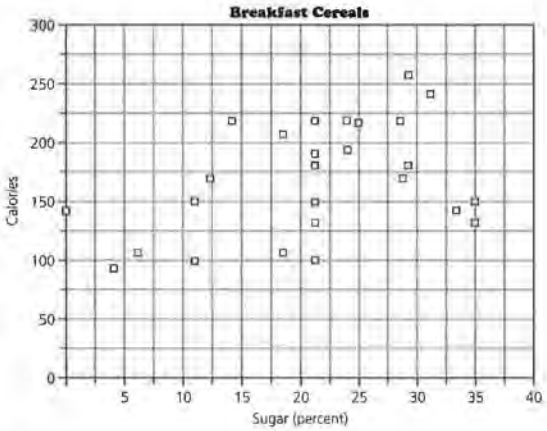
14. The plot below is of calories and percent sugar from the beginning of this lesson. An equation of the least-squares line for the number of calories, c, as a function of the percent sugar, s, is $c = 128 + 1.80s$. Use the first plot on *Activity Sheet 8* for this problem.



a. What is the slope of the equation? What does it tell you about the calories and the percent of sugar?

b. Suppose you know the equation for the least-squares line, as above. What can you anticipate about the value of r?

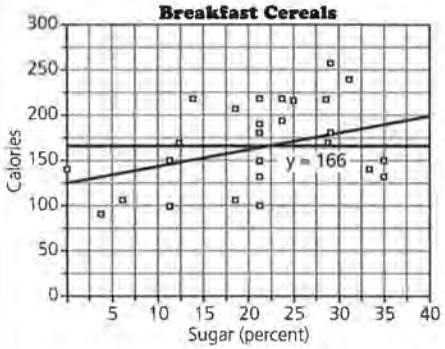
c. Sketch the least-squares line on the plot (percent sugar, calories) and the line $y = 166$ for the mean number of calories. In general, describe the difference between making predictions if you were to use the least-squares line and if you were to use the line representing the mean number of calories.

d. $r^2 = 0.132$ for the data. What is r and what does this tell you about the association between the percent of sugar and the number of calories in breakfast cereal?

14. a. The slope is 1.80, which means that increasing the sugar by 1% results in an increase of almost 2 calories.

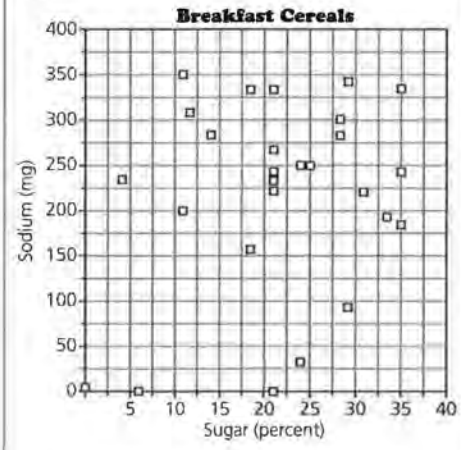
b. The least-squares equation does not tell you much about r. However, you do know that the value of r is not zero, because if it were, the line would have zero slope and you also know that r has the same sign as the slope.

c. It appears that making predictions using the least-squares line would be superior. More points are closer to the regression line than to the mean line.



d. $r = 0.363$, which indicates that there is little or weak linear association between the percent of sugar and the number of calories in breakfast cereal. About 15% of the variation in the calories is accounted for by the least-squares line.

15.



a. There does not appear to be much of a connection.

b. $r = 0.2343$; $r^2 = 0.0549$; only about 5% of the variation in the sodium can be accounted for by the least-squares line.

c. $y = 2.802x + 149.36$, so the slope is 2.802.

d. This is not true. The slope describes the change in the predicted value of one characteristic with respect to the other characteristic. The correlation coefficient expresses the degree to which the relationship between the characteristics is linear.

e. The correlation coefficient indicates that there is very little linear connection between the percent of sugar and the amount of sodium.

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15. Use the cereal data at the beginning of this lesson to plot the data for the number of milligrams of sodium as a function of the percent of sugar (percent sugar, mg sodium). Use the second grid on *Activity Sheet 8*.

a. How strongly do you think the variables are related?

b. Use a calculator to find the correlation coefficient. What is r^2 and what does it tell you?

c. Find the least-squares regression line for the data. What is the slope of the regression line?

d. Comment on this statement: *The slope of the regression line measures the same thing as the correlation coefficient.*

e. What does the correlation coefficient tell you about the relationship between the amount of sugar and the amount of sodium?

Correlation and Cause and Effect

People often confuse correlation with cause and effect. *Just because two variables are correlated does not mean that one causes the other.*

- The two variables could both be a function of some other cause,
- the supposed cause could be the effect, or
- the relationship could be purely coincidental.

Consider the relationship between overall grade-point averages and grades in English. The association may be strong, but English grades alone do not cause high grade-point averages; other courses also contribute. The association between grade-point averages and hours of study is high, and it is reasonable to assume that the time spent studying is a primary cause of grade-point averages. In some sense, however, higher grades could also cause one to study more. The correlation between grade-point averages and SAT scores is strong, but neither variable causes the other. A good SAT score does not cause high grade-point averages.

Sometimes the relationship is purely coincidental. It could be that the correlation between grade-point averages and distance from school was strong. But it seems unlikely that all the good students live the same distance from school. Much more reasonable is the assumption that the connection is coincidental, and that there is no real link between distance and grade point.

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16. Suppose you had scatter plots for data dealing with the following situations:

    Number of cigarettes smoked and number of deaths due to cancer

    Number of basketball fouls committed and number of points scored in field goals

    Years of Latin and SAT scores

    Weight of a car and the amount of fuel used

    Years of schooling and yearly income

    Foot length and reading level

    a. What do you think each graph might look like?

    b. Do you think there is positive correlation, negative correlation, or no correlation for each?

    c. For those that have fairly good correlation, does one cause the other? Explain.

17. Think of an example involving two variables with a positive correlation in which one variable does not cause the other.

18. Toy prices from a fall catalog are given in the table below.

Toy	Price ($)
Stacking rings	10.99
Curious George books (3)	8.99
Popcorn popper	14.95
Stuffed animal	18.50
Baby All Gone	19.99
Infant rocking horse	25.78
Barbie doll	21.99
Lego set	42.49

    In December, the company offered a reduced price of $3.00 off every item as part of a holiday sale.

    a. What does the plot (old, new) for the prices look like?

    b. Estimate the correlation between the new and the old prices. Calculate the correlation and compare it to your estimate.

    c. Suppose the company slashed every price by 10%. Recalculate the correlation between the new and old prices. Explain any differences.

16. a. The graph of the number of cigarettes smoked and the number of deaths due to cancer is fairly linear. The graph of the number of fouls and number of points scored in field goals is fairly linear. The graph of the years of Latin and SAT scores is likely to be linear. The graph of the weight of a car and amount of fuel used is likely to be linear. The graph of the years of schooling and yearly income might be linear or curved. The graph of foot length and reading level is not linear.

b. The graph of the number of cigarettes smoked and the number of deaths due to cancer has positive correlation. The graph of the number of fouls and number of points scored in field goals has negative correlation. The graph of the years of Latin and SAT scores has positive correlation. The graph of the weight of a car and amount of fuel used has positive correlation. The graph of the years of schooling and yearly income has positive correlation. The graph of foot length and reading level has no correlation.

c. Positive correlation:

Cigarettes and deaths, smoking causes cancer.

Latin and SAT scores, very little or no causal relationship.

Car weight and fuel, weight causes more fuel consumption.

Years in school and income, degrees may advance salary.

Negative correlation:

Fouls and points scored in field goals, getting more fouls prevents getting field goals.

17. Answers will vary; sample: The relationship between the total sales of greeting cards and of flowers is likely to be linear with a positive slope, but one does not cause the other. The correlation is caused by events like Mothers' Day.

18. a. The graph is linear.

b. The correlation is 1.

c. The correlation remains 1.

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19. The data below are the number of minutes played, number of rebounds, number of assists, and number of points made by the Chicago Bulls in the 1995–1996 NBA season in which they won their fourth NBA championship.

Chicago Bulls, 1995–1996

Players	Minutes	Rebounds	Assists	Points
Jordan	3090	543	352	2491
Pippen	2825	496	452	1496
Kukoc	2103	323	287	1065
Longley	1641	318	119	564
Kerr	1919	110	192	688
Harper	1886	213	208	594
Rodman	2088	952	160	351
Wennington	1065	174	46	376
Salley*	673	140	54	85
Buechler	740	111	56	278
Simpkins	685	156	38	216
Brown	671	66	73	185

Source: *World Almanac and Book of Facts*, 1997
* Played for more than one team

a. Find the correlation coefficient between the following pairs of variables and discuss what each coefficient tells you.

i. Number of rebounds and number of points

ii. Number of assists and number of rebounds

iii. Minutes played and number of points

iv. Minutes played and number of rebounds

b. For which pair of variables is the correlation the strongest? Do you think there is a cause-and-effect relationship?

c. For which pair of variables is the correlation the weakest?

19. a. Number-of-rebounds and number-of-points correlation is 0.405, which means that about 16% of the variation in points can be accounted for by the least-squares line.

Number-of-assists and number-of-rebounds correlation is 0.487, which means that about 23% of the variation in rebounds can be accounted for by the least-squares line.

Minutes-played and number-of-point correlation is 0.87, which means that about 75% of the variation in points can be accounted for by the least-squares line.

Minutes-played and number-of-rebounds correlation is 0.66, which means that about 43% of the variation in rebounds can be accounted for by the least-squares line.

b. The strongest correlation is between minutes played and the number of points. There is some kind of causal relationship, because if you aren't playing, you can't get any points.

c. The weakest correlation is between number of rebounds and number of points.

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20. The following quote appeared in a suburban Milwaukee newspaper article entitled "Spending More on Police Doesn't Reduce Crime."

A CNI study of crime statistics and police department budgets over the last four years reveals there really is no correlation between what a community spends on law enforcement and its crime rate.

a. Make a sketch of what you think a plot of the data above would look like.

b. Use the data in the table below about the suburban crime rate and the per-capita spending on police. Plot the data and find the correlation coefficient.

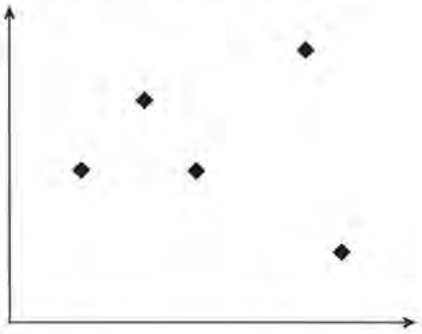
Community	Suburban Crime Rate per 1,000 Residents	Per-Capita Spending on Police ($)
Glendale	74.39	222.25
West Allis	50.43	164.47
Greendale	50.25	123.43
Greenfield	48.68	143.59
Wauwatosa	48.52	150.34
South Milwaukee	43.20	110.64
Brookfield	42.16	131.42
Cudahy	41.47	137.12
St. Francis	41.32	144.84
Shorewood	40.84	156.39
Oak Creek	40.84	160.41
Brown Deer	37.09	150.57
Germantown	31.16	125.86
Menomonee Falls	29.73	159.28
Hales Corners	27.04	155.40
New Berlin	25.45	125.33
Franklin	23.09	94.92
Elm Grove	21.86	191.64
Whitefish Bay	21.28	120.34
Muskego	17.00	105.35
Fox Point	15.07	132.85
Mequon	11.31	136.39

Source: *Hub*, November 4, 1993

c. Does the correlation coefficient support the conclusions in the paragraph?

d. What does r^2 indicate about the relationship between spending and the crime rate?

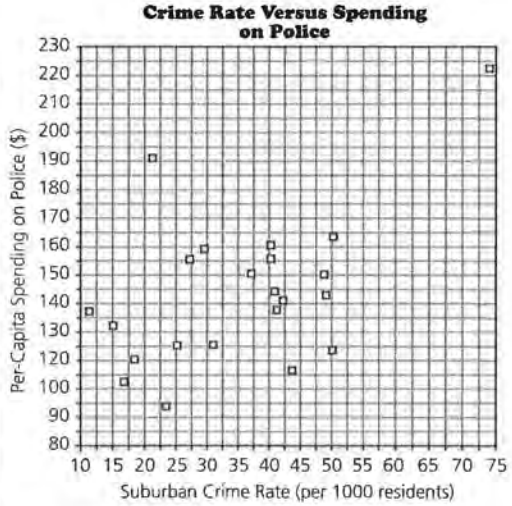
20. a. Answers will vary; sample:



b. Correlation coefficient $r = 0.501$
See graph below.

c. No; this r shows a small but distinct correlation.

d. Only 25% of the variation in crime can be accounted for by the least-squares line.



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21. It appears that they used data about the number of people who listen to country music and those who commit suicide. They plotted the percent increases in each and, evidently, it appeared linear and had an *r*-value close to 1. The problem is that the article claims a cause-and-effect relationship. While the correlation between the percent of market share and the number of suicides might be close to 1, this indicates that the plot may be nearly linear and could be represented by a straight line.

21. The following article was taken from the *Milwaukee Journal* on Friday, November 20, 1992.

Yes, But How Come Those Hee Haw People Smile?

The mournful lyrics of country music may lament that "I'm So Lonesome I Could Cry." Now a statistical study claims such songs bring out more than tears in beer—they may lead to an increase in suicide. The study, co-authored by Steven Stack of Wayne State University in Detroit and Auburn University sociologist Jim Gundlach, was published in the September issue of *Social Forces*, a journal of sociology. Gundlach said Tuesday that the survey found a correlation between suicides in America and listening to country music, known for its often plaintive sounds and themes of loss and loneliness. Gundlach said the survey was based on the radio market share for country music in the nation's 49 leading music markets and the incidence of suicides in those areas. He said the study, based on 1985 statistics, found that for every 1% increase in country music's share of the market, there was a corresponding increase in the number of suicides.

Comment on the use of the statistics in this article. Describe the kind of data used, what a plot might look like, and what the various terms mean.

LESSON 11

Which Model When?

Materials: graph paper, rulers
Technology: graphing calculator or computer spreadsheet program
Pacing: 2–3 class periods and homework

Overview

In the previous lessons, students learned how to find the *best* linear relationship between two variables, called the least-squares regression line. In this lesson, we will investigate if the least-squares regression line is always the best line. If we decide that it is *not* the best line, what are the factors influencing our decision? There are other linear models that can be used to fit a line to data; one such line is the median-fit line. In this lesson, students will compare the least-squares line to the median-fit line, make decisions as to which model is better, and investigate the circumstances that determine when each is appropriate.

Teaching Notes

This lesson will require a knowledge of the median-fit line. If the students have not been exposed to the median-fit line, Problem 2 can be eliminated, as well as Problems 5b, part of 6a, 6c, 8, 9, and 10. Problems 8, 9, and 10 could be included if you simply ask what effect eliminating the outlier would have on the graph in each case and why.

Note: Information on the median-fit line is covered in *Exploring Data,* a booklet in the *Quantitative Literacy Series* published by Dale Seymour.

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LESSON 11

Which Model When?

Is the least-squares regression line always the *best* line to use?

What considerations should be kept in mind when using the least-squares model?

OBJECTIVE

Recognize the need for different linear models and the impact of outliers on the least-squares regression line.

INVESTIGATE

In some earlier work, the median-fit line or another model may have been used for summarizing the relationship between two variables. This lesson centers on using the least-squares regression line to describe that relationship.

Discussion and Practice

The following table lists team members of the 1997 NFC champion Green Bay Packers. It gives position, height, weight, and body-mass index for each player.

Player	Position	Height (ft-in.)	Weight (lb)	Body-Mass Index
Robert Brooks	WR	6-0	180	24.5
Ryan Longwell	K	6-0	185	25.1
Craig Hentrich	P	6-3	200	25.1
Doug Evans	CB	6-1	190	25.1
Bill Schroeder	WR	6-2	198	25.5
Antonio Freeman	WR	6-1	194	25.6
Steve Bono	QB	6-4	212	25.9
Don Beebe	WR	5-11	185	25.9
Darren Sharper	CB/S	6-2	205	26.4
Eugene Robinson	S	6-0	197	26.8
Roderick Mullen	CB/S	6-1	204	27.0
Doug Pederson	QB	6-3	216	27.1
LeRoy Butler	S	6-0	200	27.2
Tyrone Williams	CB	5-11	195	27.3
Terry Mickens	WR	6-0	201	27.3
Mike Prior	S	6-0	203	27.6

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Player	Position	Height (ft-in.)	Weight (lb)	Body-Mass Index
Derrick Mayes	WR	6-0	205	27.9
Chris Darkins	RB	6-0	210	28.5
Brett Favre	QB	6-2	225	28.9
Aaron Hayden	RB	6-0	216	29.4
Jeff Thomason	TE	6-5	250	29.7
Travis Jervey	RB	6-0	222	30.2
Mark Chmura	TE	6-5	253	30.1
Dorsey Levens	RB	6-1	230	30.4
Tyrone Davis	TE	6-4	255	31.1
Lamont Hollinquest	LB	6-3	250	31.3
Seth Joyner	LB	6-2	245	31.5
Brian Williams	LB	6-1	240	31.7
Paul Frase	DE	6-5	267	31.7
Bernardo Harris	LB	6-2	247	31.8
Keith McKenzie	LB/DE	6-3	255	31.9
George Koonce	LB	6-1	243	32.1
William Henderson	FB	6-1	249	32.9
Rob Davis	LS	6-3	271	33.9
Santana Dotson	DT	6-5	285	33.9
John Michels	T	6-7	304	34.3
Jeff Dellenbach	C/G	6-6	300	34.7
Gabe Wilkins	DE	6-5	295	35.1
Marco Rivera	G	6-4	295	36.0
Adam Timmerman	G	6-4	295	36.0
Bob Kuberski	DT	6-4	295	36.0
Reggie White	DE	6-5	304	36.1
Jermaine Smith	DT	6-3	289	36.2
Ross Verba	G/T	6-4	299	36.5
Bruce Wilkerson	T	6-5	310	36.8
Aaron Taylor	G	6-4	305	37.2
Frank Winters	C	6-3	300	37.6
Darius Holland	DT	6-5	320	38.0
Earl Dotson	T	6-4	315	38.4
Joe Andruzzi	G	6-3	313	39.2
Gilbert Brown	DT	6-2	345	44.4

Source: *Milwaukee Journal Sentinel*, November 18,1997

Solution Key

Discussion and Practice

1. **a.** The line is higher on the right than it would have been if this point were not present.

b. The predictions would be quite accurate.

c. The sum of the squared errors is 63.59, and the root mean squared error is 1.116.

d. 0.0912(140) + 6.8401 = 20.7; the root mean squared error means the actual BMI likely lies between 20.7 – 1.116 and 20.7 + 1.116, that is, between 19.584 and 21.816.

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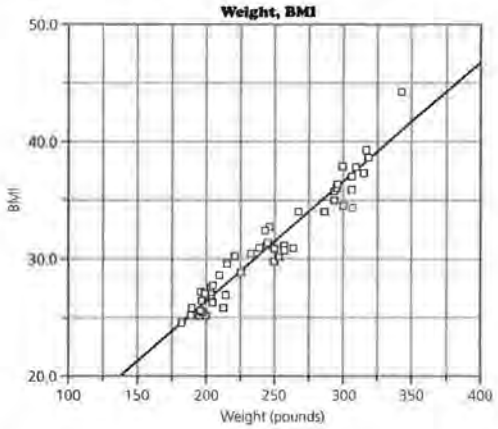
Body-mass index, or *BMI*, is a commonly used guideline to gauge obesity and is based on height and weight. A BMI of 27.3 or above for women and 27.8 or above for men are considered obese. To calculate BMI, you must first convert your weight and height into kilograms and meters. Divide weight in pounds by 2.2 for weight in kilograms. Multiply height in inches by 0.0254 for height in meters. BMI is weight in kilograms divided by (height in meters) squared. A plot of the weight and BMI with the least-squares regression line is shown below.



1. Use the data about the BMI for each of the following problems.

a. One player weighs more than all others and has a high BMI. How does the line seem to reflect the impact of that point?

b. The correlation coefficient, r, is 0.971. What does this tell you about predicting the BMI from the weight?

c. The equation of the line is BMI = 0.09884 × weight + 6.8401. Find the sum of the squared errors and use it to find the root mean squared error.

d. Use your information to predict the BMI for a person that weighs 140 pounds. What effect will the error have on your prediction?

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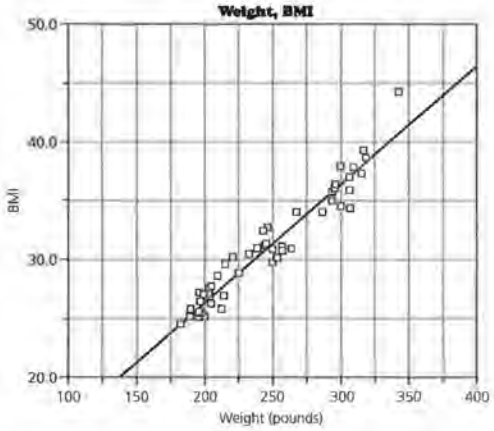
2. A median-fit line has been plotted for the data in the graph below. The equation of the median-fit line is BMI = 0.098 × weight + 6.898.



   a. How does the graph of the median-fit line differ from the graph of the least-squares regression line?

   b. Find a measure of the typical error in prediction using the median-fit line. How do you define *typical* error?

   c. Use the median-fit line to identify the BMI for a player who weighs 140 pounds.

   d. Which line do you think will be a better predictor, the median-fit line or the least-squares line? Justify your answer.

3. Now delete the point that seems to be an outlier and analyze the data again.

   a. Find an equation of the least-squares line without using that point.

   b. What is the new correlation? What does that tell you about predicting the BMI from the weight?

   c. How does the new equation compare to the equations from Problems 1 and 2?

   d. Do you think removing a point before you do your analysis is justified? Explain your answer.

2. a. The regression line is steeper because of the greater slope. However, since the difference in the slope is only 0.00112 and the intercepts are also very close, the lines appear to be identical.

b. Students may have different views of what is the *typical* error. The sum of the squared errors is 65.42, and the root mean squared error is 1.13.

c. 0.098(140) + 6.898 = 20.618

d. In this case, it is very difficult to choose. The outlier has little effect and the lines are so close that it really makes very little difference which equation you use.

3. a. BMI = 0.09561 × weight + 7.598

b. The correlation coefficient is $r = 0.9723$, which indicates a high level of accuracy in the predictions.

c. The new equation is BMI = 0.09561 × weight + 7.598. The first regression equation is BMI = 0.09912 × weight + 6.8401. The median-fit equation is BMI = 0.09800 × weight + 6.898.

The slope of the new line is less, so the line is flatter than the other two. It is also higher on the left and lower on the right than the other two.

d. The new line fits the data better visually, its sum of squared errors is less, its smaller root mean squared error is less, and it has a greater correlation coefficient, all of which lend support to the removal of the point.

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4. The four data sets below were constructed by Frank Anscombe ("Graphs in Statistical Analysis," *The American Statistician*, February, 1973). Complete parts a and b for each set. Then complete part c.

   a. Plot the data. Plot x on the horizontal axis and y on the vertical axis.

   b. Find the least-squares regression line and the correlation coefficient.

   c. What conclusions can you make after you have investigated all four data sets?

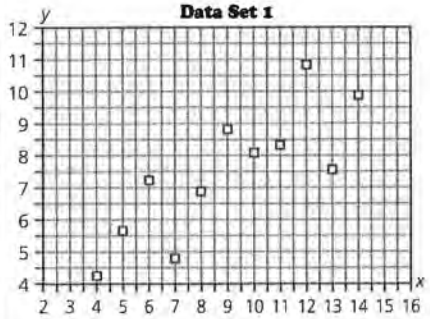
Data Set 1		Data Set 2		Data Set 3		Data Set 4	
x	y	x	y	x	y	x	y
10	8.04	10	9.14	10	7.46	8	6.58
8	6.95	8	8.14	8	6.77	8	5.76
13	7.58	13	8.74	13	12.74	8	7.71
9	8.81	9	8.77	9	7.11	8	8.84
11	8.33	11	9.26	11	7.81	8	8.47
14	9.96	14	8.10	14	8.84	8	7.04
6	7.24	6	6.13	6	6.08	8	5.25
4	4.26	4	3.10	4	5.39	9	12.50
12	10.84	12	9.13	12	8.15	8	5.56
7	4.82	7	7.26	7	6.42	8	7.91
5	5.68	5	4.74	5	5.73	8	6.89

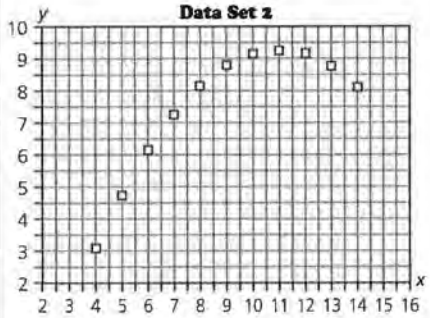
Source: *Statistics for Business: Data Analysis and Modeling*, Jonathan D. Cryer/ Robert B. Miller

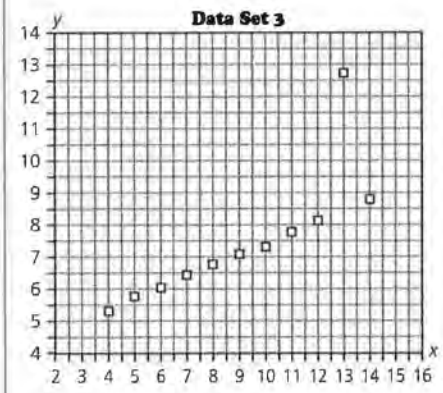
Summary

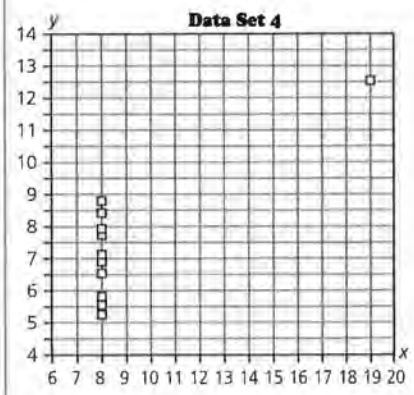
Finding a good model for a set of data involves much more than *number crunching*. It is important that you use all of the information, including a scatter plot of the data, to find a model that represents the data well. Both the least-squares regression line and the correlation coefficient are sensitive to extreme values. Patterns that are not linear are not captured by the correlation coefficient. Some nonlinear patterns have a high correlation. This makes it even more important for you to study a scatter plot of the data as a first step in your analysis.

4. a.









b. Data Set 1: $y = 0.5x + 3$; $r = 0.8164$

Data Set 2: $y = 0.5x + 3$; $r = 0.8162$

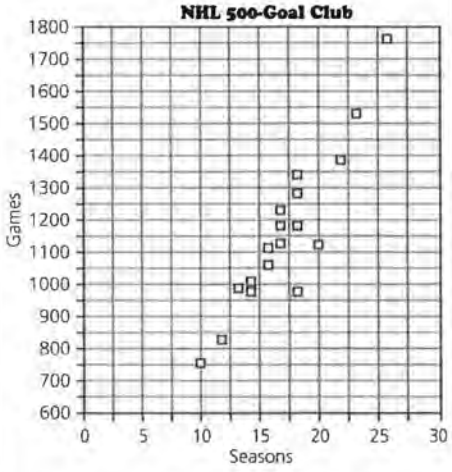
Data Set 3: $y = 0.5x + 3$; $r = 0.8163$

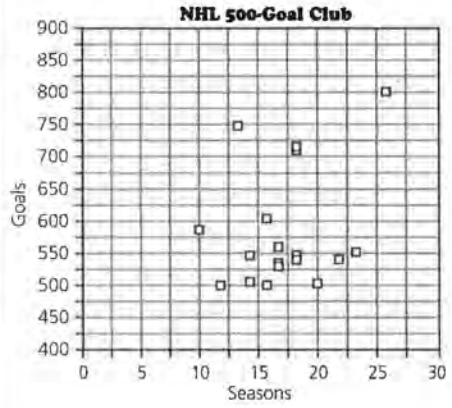
Data Set 4: $y = 0.5x + 3$; $r = 0.8165$

c. These four data sets demonstrate that the equation of the line and the value of r do not determine the shape of the plot. All four have very different shapes, but the same equations and r-values.

Practice and Applications

5.





a. There is a linear relationship only between seasons and games: $G = 57.58S + 173.4$. Using this line, one would expect a 500-goal player who played 19 seasons to play in 1,267 games.

b. The least-squares line is appropriate because there are no particular outliers. This line seems to predict well when the plot is examined with the line overlaid.

c. The model predicts that he will play 1,786 games. Answers will vary about the reliability of this prediction. This may be high, because one would expect that the longer you play, the more injuries and the more missed games.

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Practice and Applications

Many times there are relationships between variables involved in sports situations. The following is from an article in *USA Today*.

Center Admitted to an Elite Group

The National Hockey League's 500-goal club added its 18th member Saturday night when Los Angeles Kings center Jari Kurri scored an empty-net goal in an 8–6 victory against the Boston Bruins. The list includes three "active" players.

Player	Primary or Current Team	Seasons	Games	Goals
Gordie Howe	Detroit	26	1767	801
Wayne Gretzky*	Los Angeles	13	999	749
Marcel Dionee	Los Angeles	18	1348	731
Phil Esposito	Boston	18	1282	717
Bobby Hull	Chicago	16	1063	610
Mike Bossy	N.Y. Islanders	10	752	573
Guy Lafleur	Montreal	17	1126	560
John Bucyk	Boston	23	1540	556
Maurice Richard	Montreal	18	978	544
Mike Gartner	N.Y. Rangers	14	1011	542
Stan Mikita	Chicago	22	1394	541
Frank Mahovlich	Toronto	18	1181	533
Bryan Trottier	N.Y. Islanders	17	1238	520
Gil Perreault	Buffalo	17	1191	512
Michel Goulet*	Chicago	14	976	511
Jean Beliveau	Montreal	20	1125	507
Lanny McDonald	Toronto	16	1111	500
Jari Kurri *	Los Angeles	12	833	500

* Active player Source: *USA Today*

5. Make scatter plots of (seasons, games) and (seasons, goals).

a. If there is a linear relationship in either plot, find a model and use it to predict the number of games or goals for a future hockey player who scores at least 500 goals and plays for 19 seasons.

b. Which model did you use and why? How well do you think your model will predict?

c. Wayne Gretzky was still an active player at the time of this article. Use your model to predict how many games or goals he will have if he plays for 28 seasons. Do you think this is reasonable?

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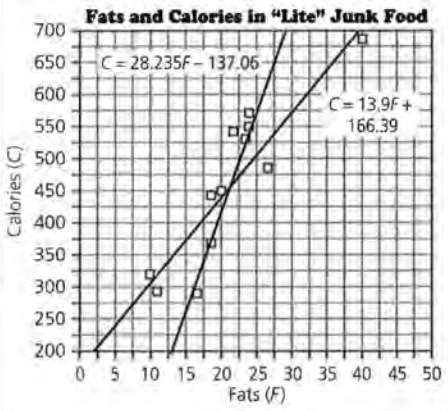
6. a.



An equation of the median-fit line is $C = 28.235F - 137.06$. An equation of the least-squares line is $C = 13.9F + 166.39$.

The lines look quite different. The least-squares line has a much less steep slope because of the apparent outlier.

b. $r = 0.8898$, which means that almost 80% of the variation in the number of calories is explained by the least-squares regression line.

c. The median-fit line looks like it may be a better fit, as it seems to lie closer to more of the points.

d. The median-fit line predicts 710 calories, and the least-squares line predicts 583.39 calories. It may be that the least-squares line is a better predictor for this point, because this value is near the extreme, like the outlier, rather than being within the main clump of the data.

6. The following data came from a poster on the door in a school lunchroom.

"LITE" JUNK FOOD ... How Healthy Is It Really?

Item	Fats	Calories	Calories from Fat	Percent of Calories from Fat
McDonald's				
McLean Deluxe Sandwich	10	320	90	28.13%
Filet O' Fish Sandwich	18	370	162	43.78%
McLean Deluxe and Small Fries	22	540	198	36.67%
Burger King				
Weight Watchers Fettucini Broiled Chicken	11	298	99	33.23%
Fried Chicken Sandwich	40	685	360	52.56%
Wendy's				
Chicken Sandwich	20	450	180	40.00%
Baked Potato with Broccoli & Cheese	24	550	216	39.28%
Taco Salad	23	530	216	39.28%
Kentucky Fried Chicken				
Skinfree Crispy Breast	17	293	153	52.22%
Chicken Sandwich	27	482	243	50.41%
Hardee's				
Oat Bran Muffin	18	440	162	36.82%
Real Lean Deluxe and Small Fries	24	570	116	37.90%

Source: *Health & Healing*

a. Make a scatter plot of (fats, calories). Find both a median-fit line and a least-squares regression line for the data. How do the two lines compare?

b. Find the correlation coefficient. What does this tell you about the relationship between fats and calories in "lite" junk food?

c. Which model seems to summarize the relationship better?

d. Predict how many calories would be in a "lite" food that has 30 grams of fat. How well do you think your line predicts? Why?

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7. Use the data in Problem 6 for the following.
   a. What do you think the plot of calories and calories from fat should look like? Plot the data and find a line to summarize the relationship.
   b. Find the correlation coefficient.
   c. If you haven't already done so, look closely at the plot. There is something unusual about the data. Can you find a reasonable explanation for this observation?
   d. Based on the conclusions you drew in part c above, how would you adjust your model?

In Problems 8–10, plot each set of data. Decide whether a median-fit line or a least-squares regression line will give a better description of the relationship between the variables. Explain how you made your choice and why you selected the one you did.

8. The amount of waste for glass and plastic in the United States since 1960 and projected until 2010 is given in the table. Plot (glass, plastic).

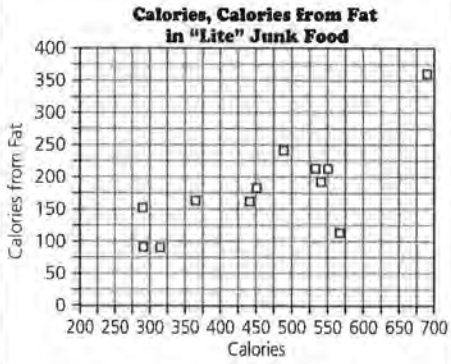
Year	Glass (million tons)	Plastic (million tons)
1960	49	94
1970	70	124
1980	80	150
1990	96	172
2000	115	200
2010	125	230

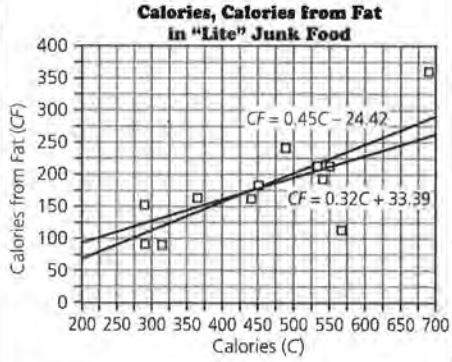
Source: *World Almanac and Book of Facts*, 1992

9. The following data are the overall scores and the price of VCRs as rated in *Consumer Reports*, October, 1997. Plot (price, score).

Brand and Model	Price ($)	Overall Score
Sony SLV-775HF	300	82
Panasonic PV-7662	290	75
Samsung VR8807	220	75
Toshiba M-683	250	75
Hitachi VT-FX624A	270	70
Sharp VC-H978U	230	68
RCA VR626FH	290	68
JVC HR-VP644U	260	65
RCA VR631HF	230	65
Mitsubishi HS-U580	380	62
Quasar VHQ760	170	60
GE VG4261	180	50
Philips Magnavox VRX362AT	240	45

7. a. Answers will vary; one might expect a quite linear relationship.





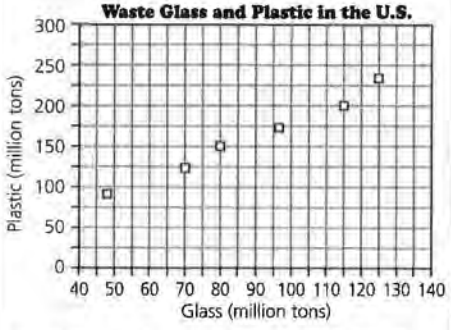
The least-squares line is $CF = 0.45C - 24.42$. The median-fit line is $CF = 0.32C + 33.39$.

b. $r = 0.75$

c. There is an obvious outlier affecting the least-squares line. The item is the Fried Chicken Sandwich, which could also contain the chicken skin, accounting for the high fat content.

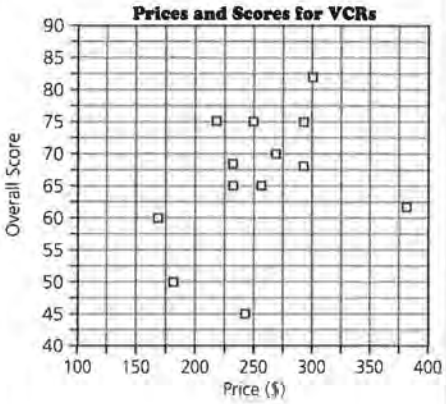
d. With the outlier removed, the regression equation becomes $CF = 0.28C + 43.20$, which is much closer to the median-fit line.

8. $P = 1.74G + 6.80$; the least-squares regression line is appropriate because there are no particular outliers.



9. The median-fit line is $S = 0.087P + 45.76$, the regression line is $S = 0.064P + 49.85$, and $r = 0.34$.

The data are clearly not linear, so it is unreasonable to fit either line.



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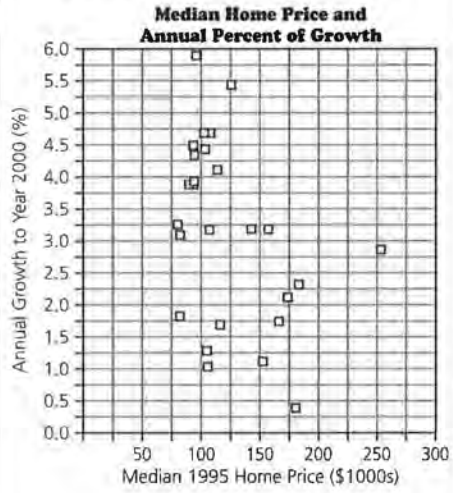
10. The median-fit line is $G = -0.0196P + 5.869$, the least-squares line is $G = -0.014P + 5.004$, and $r = -0.432$.

The data are clearly not linear, so it is unreasonable to fit either line.



10. The following list gives median 1995 prices of homes in metropolitan areas of the United States along with estimated percents of annual growth of these prices. Plot (price, growth).

Metro Area	Median Home Price 1995 ($1000s)	Annual Growth to Year 2000 (percent)
Atlanta	97.7	4.3
Baltimore	111.4	1.3
Boston	178.2	2.3
Chicago	147.4	3.2
Cincinnati	102.6	4.7
Cleveland	103.3	4.4
Dallas	96.3	3.9
Denver	126.2	5.4
Detroit	98.5	5.9
Houston	79.4	3.1
Kansas City	91.1	3.8
Los Angeles	176.9	0.4
Miami	106.5	3.2
Milwaukee	114.3	4.1
Minneapolis	107.3	4.7
New York City	169.6	1.7
Philadelphia	117.0	1.7
Phoenix	96.3	4.5
Pittsburgh	80.6	1.8
San Francisco	255.3	2.8
San Diego	172.0	2.1
Seattle	158.5	3.2
St. Louis	87.4	3.8
Tampa-St. Petersburg	77.8	3.2
Washington, D.C.	155.8	1.1

Source: *Consumer Reports*, May 1996

Teacher Resources

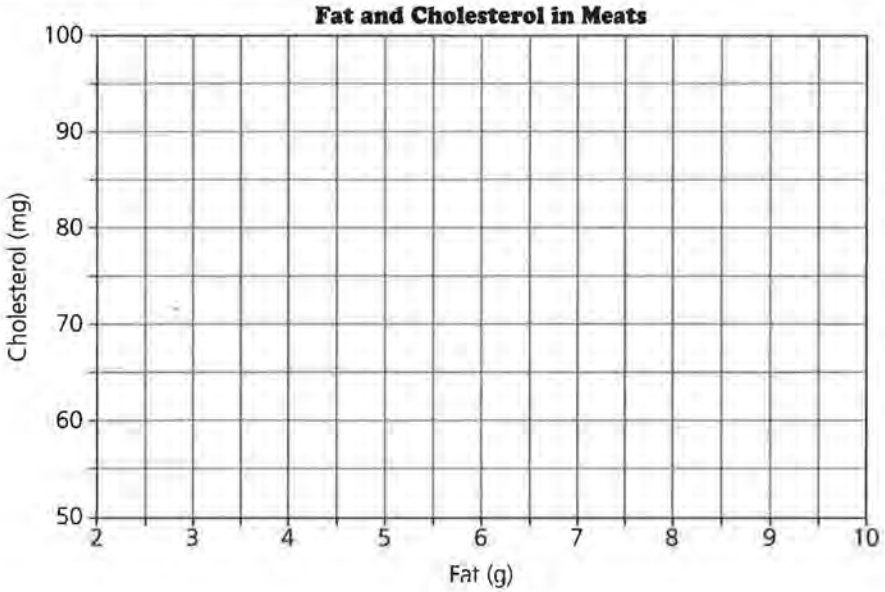
LESSON 2 QUIZ

NAME _____

The following figures for three-ounce portions of well-trimmed, cooked meat appeared in an *American Health* magazine.

Meat	Fat (g)	Cholesterol (mg)
Veal (roasted)	2.9	88
Lamb (braised)	5.1	89
Pork tenderloin (roasted)	4.1	67
Pork loin chop (broiled)	5.7	78
Pork roast	6.4	66
Chicken breast, skinned (roasted)	3.0	72
Drumstick, skinned (roasted)	4.8	79
Drumstick with skin (roasted)	9.5	77
Breast with skin (roasted)	6.6	71
Roast beef (eye of round)	4.8	59

1. Plot (fat, cholesterol). What does the plot tell you about the relationship between the amount of fat and the amount of cholesterol in the list of meats?



2. Find an equation of a line that can be used to summarize the data.

3. Use your line to predict the amount of cholesterol in a meat item that has 6.4 grams of fat. How far off was your prediction?

4. Find the residuals for the line and fill in the values in the table.

Meat	Fat (g)	Cholesterol (mg)	Residuals
Veal (roasted)	2.9	88	_____
Lamb (braised)	5.1	89	_____
Pork tenderloin (roasted)	4.1	67	_____
Pork loin chop (broiled)	5.7	78	_____
Pork roast	6.4	66	_____
Chicken breast, skinned (roasted)	3.0	72	_____
Drumstick, skinned (roasted)	4.8	79	_____
Drumstick with skin (roasted)	9.5	77	_____
Breast with skin (roasted)	6.6	71	_____
Roast beef (eye of round)	4.8	59	_____

a. What is the sum of the squared residuals?

b. What is the sum of the absolute value of the residuals?

5. Find the root mean squared error and indicate what this tells you about the prediction.

LESSON 8 QUIZ

NAME _____

1. Compare the graph of a parabola to that of an absolute-value function.

2. How is a parabola related to the least-squares linear regression line?

3. a. What are residuals?
   b. What do negative residuals represent?
   c. What problems will negatives cause in the study of residuals if they are not eliminated?

4. Let $y = 2x^2 + 3x - 40$.
   a. What are the intercepts of the graph of this equation?
   b. Describe two ways to find the vertex.

5. Use graphs and algebra to comment on the following statements.
   a. The sum of two absolute-value functions is a single absolute-value function.
   b. The sum of two quadratic functions is a single quadratic function.
   c. The sum of two linear functions is a linear function.

6. Any line is determined by a point and a slope.
   a. What point is used in determining the least-squares regression line?
   b. Describe how to find the slope of the least-squares regression line.

7. What advantage is there in using the least-squares linear-regression line as a linear model?

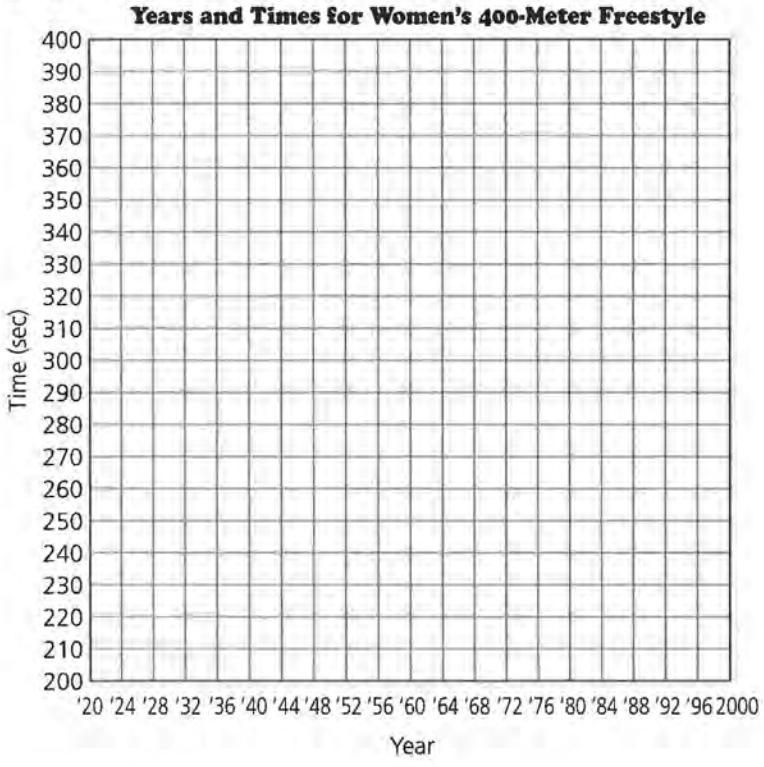
LESSON 9 QUIZ

NAME _____

The following data are the winning times for the women's 400-meter freestyle swim for the Olympics, 1924–1992.

Year	Time (min:sec)
1924	6:02.2
1928	5:42.8
1932	5:28.5
1936	5.26.4
1948	5:17.8
1952	5:12.1
1956	4:54.6
1960	4:50.6
1964	4:43.3
1968	4:31.3
1972	4:19.0
1976	4:09.9
1980	4:08.8
1984	4:07.1
1988	4:03.9
1992	4:07.2

1. Plot (year, time). If you use a calculator, sketch your graph below.



2. What is an equation of the line of best fit for this data? Draw the line on the graph above.

3. Tell what the slope of this line means.

4. What is significant about this equation?

5. Use your equation to predict the winning time for the 400-meter freestyle in the 1996 Summer Olympic Games. How confident are you of your prediction?

6. If the Olympics had been held in 1940, what would you predict the time might have been?

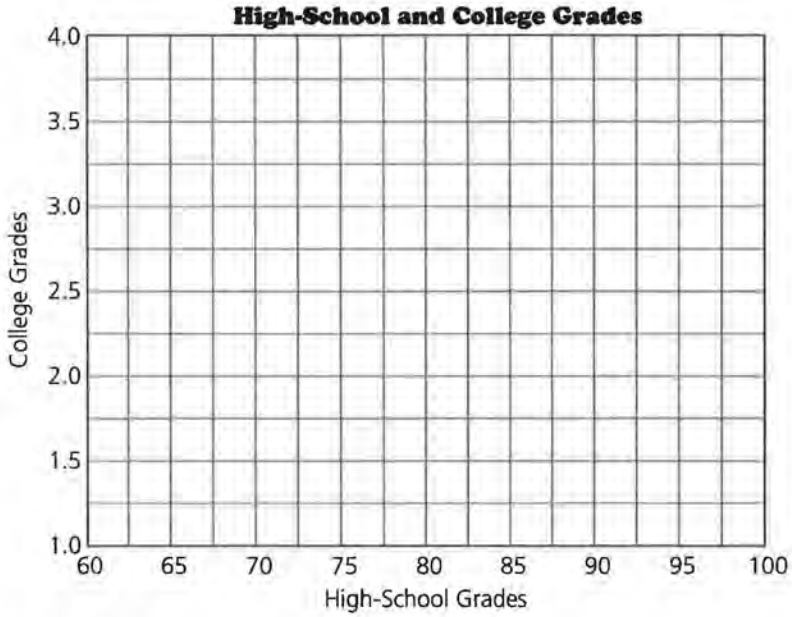
7. Summarize the conclusions you can draw from your graph and calculations.

NAME _____

Use the following data for the problems in this quiz.

High-school grades	70	92	80	73	84	77	75	85	93
College grades	1.5	3.1	2.75	2.6	3.15	2.05	2.4	2.5	2.2

1. Graph the data.



2. **a.** What kind of correlation exists between the two sets of grades?

b. What is r^2? What does this tell you about the association between these college and high-school grades?

c. What are some other factors that might be related to college grades?

3. **a.** Find the least-squares linear-regression line.

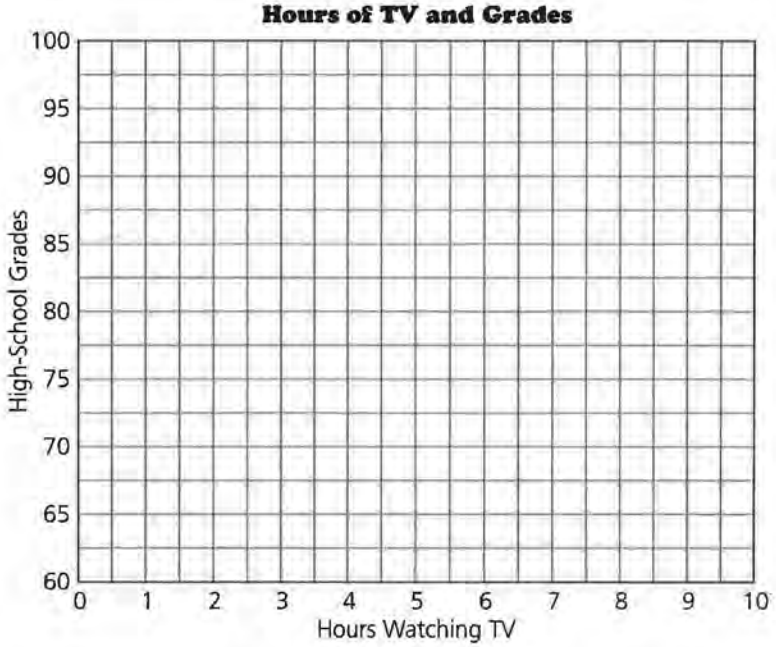
b. What is the slope and what does it tell you about the grades?

c. If a student has a high-school grade-point average of 82, predict a college grade-point average for that student. Show how you got your answer.

d. If a student is predicted to have a college grade-point average of 2.25, what was that student's high-school grade-point average?

4. The correlation between the number of hours a student watches TV and high-school grades is nearly zero.

a. Sketch a scatter plot that might represent these data.



b. Suppose you had to predict someone's high-school grade-point average knowing these data. What would you do?

c. What is the difference between a negative correlation and a correlation of zero?

5. The following was taken from an article, "Creative Classes Are Bringing Latin Back."

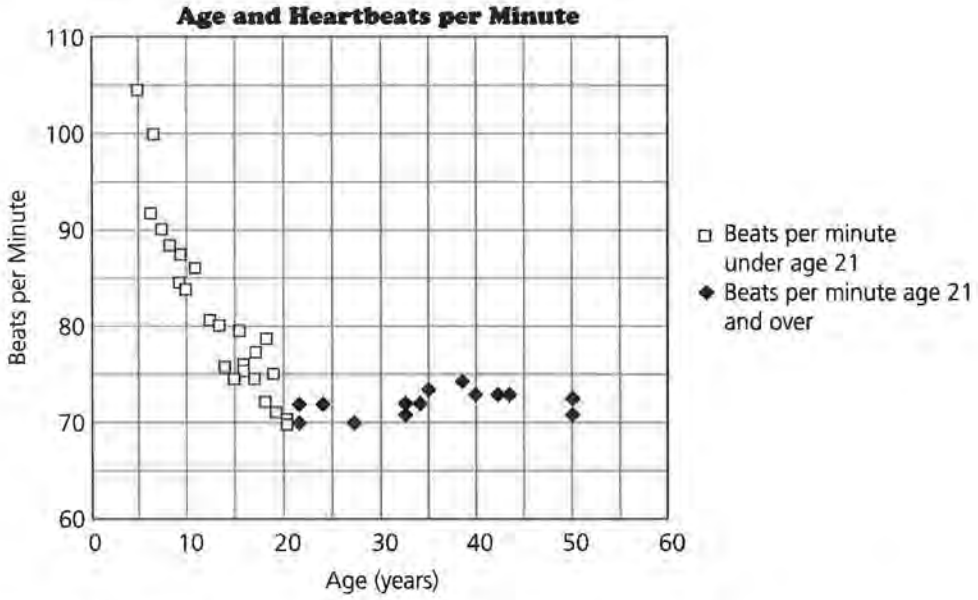
> *"There has always been a sort of elitism associated with Latin. We have really fought to combat that image," says Sally Davis, an Arlington, VA, Latin teacher and author of a 1991 report on Latin education commissioned by the American Classical League, an organization of Latin and Greek teachers.*
>
> *Still, teachers say many take Latin because research shows those students tend to get high verbal scores on the SATs.*
>
> *"Our society wants immediate payoffs for everything. This is a payoff for this."*
>
> Source: *USA Today,* by Mary Beth Marklein

Comment on this article in terms of the statistics you have learned.

NAME _____

If you do your work on a calculator, indicate what you entered and show the results. Make a sketch of any graph you use. Be sure to show enough so that I can understand what you did to find your answers. Some of the questions do not prescribe a specific technique. Use the most appropriate technique you know from those we have studied and tell what you are doing and why. You do not have to waste time being particularly neat.

1. Consider the plot below of the number of heartbeats per minute as a function of age.



a. How many heartbeats per minute would you expect for someone 17 years old?

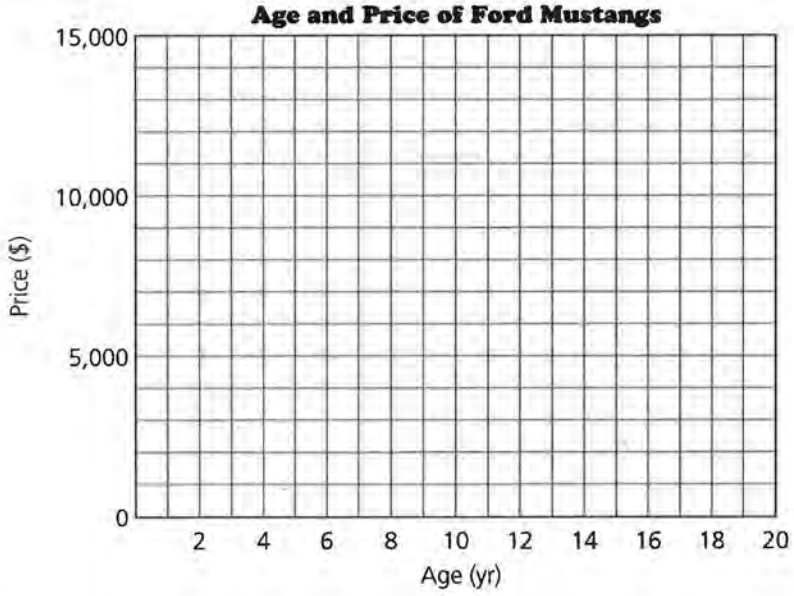
b. Describe the relationship you see in the plot between age and the number of heartbeats per minute.

c. What is a typical number of heartbeats per minute for someone who is older? How can you see this from the plot?

2. These prices of used Ford Mustangs were published in the *Milwaukee Journal* on November 21, 1993. Plot (age, price). The age was calculated as if a 1993 car were 1 year old.

Year	Age (years)	Price ($)
1980	14	900
1980	14	575
1983	11	1,950
1985	9	2,695
1986	8	1,795
1986	8	1,695
1986	8	2,500
1987	7	7,495
1988	6	4,300
1988	6	6,800
1988	6	8,495
1990	4	7,295
1990	4	11,995
1993	1	8,493
1993	1	12,799

a. Plot the data and the least-squares linear-regression line.



b. What does the slope represent? What do the intercepts represent?

c. Determine the residuals for the regression line. What is the sum of the squared residuals?

Toni used the line $y = -500x + \$9{,}000$ to summarize the relationship. Which has the smallest sum of squared residuals, Toni's equation or the equation from part a? How do you know?

3. Consider the following data set that lists the price and the ratings for CD players taken from *Consumer Reports*, March, 1990.

Price ($)	Rating Score
310	97
373	96
385	94
368	94
363	93
316	93
400	93
367	93
350	92
250	82
245	80
425	79
263	90
306	90
290	89
230	87
195	86
168	82

a. Plot the data (price, rating score) and describe the plot.

b. What is the correlation coefficient and what does it tell you?

c. What might make the correlation coefficient higher?

d. What percent of the rating score can be predicted by knowing the price? What other factors might contribute to a high rating?

e. Find the least-squares linear-regression line.

f. Prove that the centroid lies on the least-squares linear-regression line.

4. Tell what is wrong with each of the following statements.

   a. A psychologist speaking to a meeting of the American Association of University Professors said, "The evidence suggests that there is nearly correlation zero between teaching ability of a faculty member and his or her research productivity."

      The student newspaper reported this as, "Professor McDaniel said that good teachers tend to be poor researchers, and good researchers tend to be poor teachers."

      Source: *Statistics Concepts and Controversies,* David S. Moore, Second Edition

   b. "Education as a case for beer tax." There is a strong correlation between graduation rates for kids leaving college and state beer taxes. A Duke study found that the portion of kids graduating from college rose from 15% to 21% when the beer tax jumped from 10 cents to $1 a case.

      So the article states—"Sure studying helps. But if you want to improve college graduation rates, you could also increase taxes on a case of beer."

      Source: *The Milwaukee Journal*

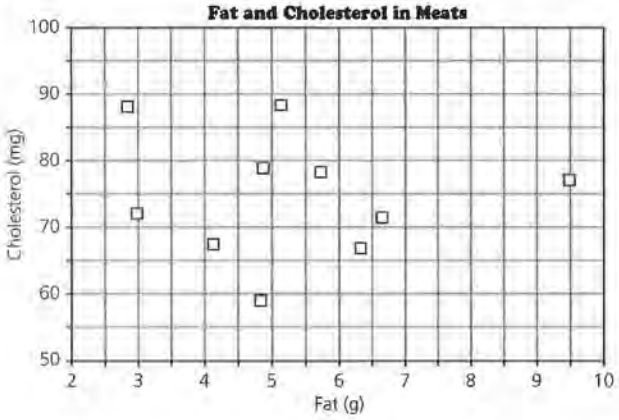
5. Define each and write a response to the question.

   a. Residual; how is it used in statistics?

   b. Correlation; why is it important?

   c. Least-squares regression line; what does it do and why is it important?

1. The graph shows there is very little linear association between the two variables.



2. Sample equation: cholesterol = $-0.50524 \times$ fat + 77.273.

3. Substituting into the formula yields 74 mg cholesterol. The table shows a value of 66 mg of cholesterol for 6.4 g of fat; therefore the difference is 8 mg of cholesterol.

4.

Meat	Fat (g)	Cholesterol (mg)	Residuals
Veal (roasted)	2.9	88	12.193
Lamb (braised)	5.1	89	14.304
Pork tenderloin (roasted)	4.1	67	–8.201
Pork loin chop (broiled)	5.7	78	3.6071
Pork roast	6.4	66	–8.039
Chicken breast, skinned (roasted)	3.0	72	–3.757
Drumstick, skinned (roasted)	4.8	79	4.1525
Drumstick with skin (roasted)	9.5	77	4.5269
Breast with skin (roasted)	6.6	71	–2.938
Roast beef (eye of round)	4.8	59	–15.85

a. The sum of the squared residuals is 809.7913.

b. The sum of the absolute value of the residuals is 77.566.

5. The root mean squared error is 8.999.
It means that, on average, the value predicted by using the linear equation is off by ± 8.999 mg.

1. The absolute-value graph is made up of two rays and is in the shape of a V unless it is composite. The graph of a parabola is a smooth U-shaped curve.

2. Since the sum of absolute-value functions does not always have a determinable minimum point and the sum of quadratic functions always will have a determinable minimum point, the quadratic function, or parabola when graphed, is used to determine the least-squares regression line.

3. a. A residual is the value of the difference between the observed value of the data for a given x-value and the predicted value of the data at the same x-value.

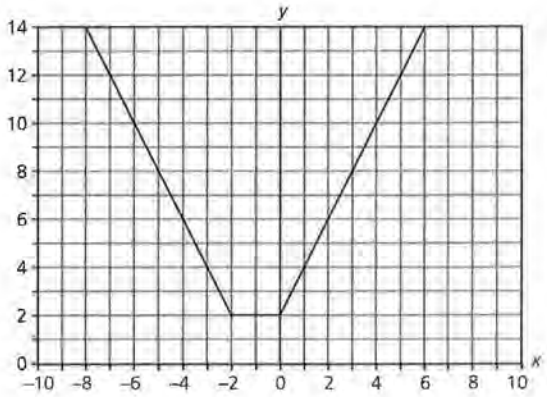
b. Negative residuals indicate that the predicted value is too large.

c. If negative and positive values cancel each other out, the residual sum may be close to zero and the line may seem to be a better fit than it really is. To avoid this situation, either the absolute value of the residuals could be summed or the residuals could be squared and then summed. In each case, the lesser number indicates the best line.

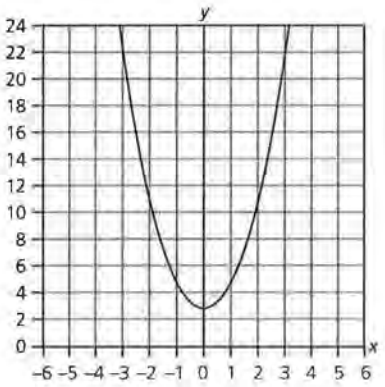
4. a. The x-intercepts are (3.79, 0) and (–5.29, 0). The y-intercept is (0, –40).

b. The vertex may be found by using the form $y = a(x - h)^2 + k$ or the formula $x = -\frac{b}{2a}$.

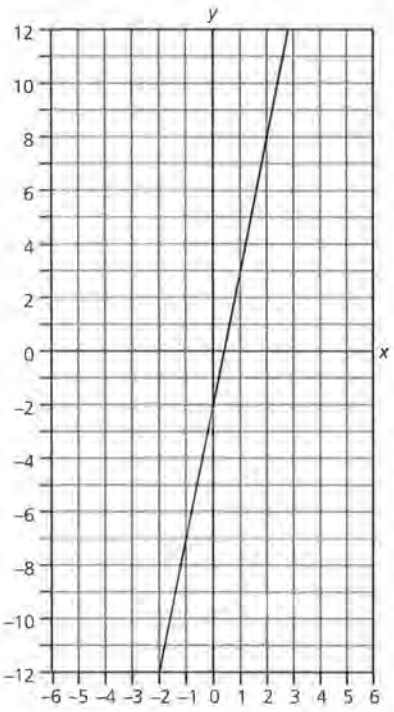
5. a. Adding two absolute-value functions has no algebraic means of simplification; however, it is not possible to introduce another function, as well. Therefore, the sum of two or more absolute functions will have to remain an absolute-value function. Graphically, the sum of two absolute value functions does take on a different appearance. Consider, for example, $y = |x| + |x + 2|$, graphed at the right.



b. The algebraic method of summing quadratic functions is to sum terms of like degree. This method will assure that the sum is also a quadratic function, as long as the terms of degree 2 do have exact opposite numerical coefficients. Consider, for example, $y = x^2 + (x^2 + 3)$, graphed below.



c. The algebraic method of adding two linear functions is to add like terms ensuring that unless the two variable terms have the exact opposite numerical coefficient the sum of two linear function is another linear function. Consider, for example, $y = (3x + 4) + (2x - 6)$, graphed below.



6. a. The point used to determine the least squares regression line is the centroid $(\bar{x}, \bar{y})$, that is, the mean of the x-values and the mean of the y-values of the data.

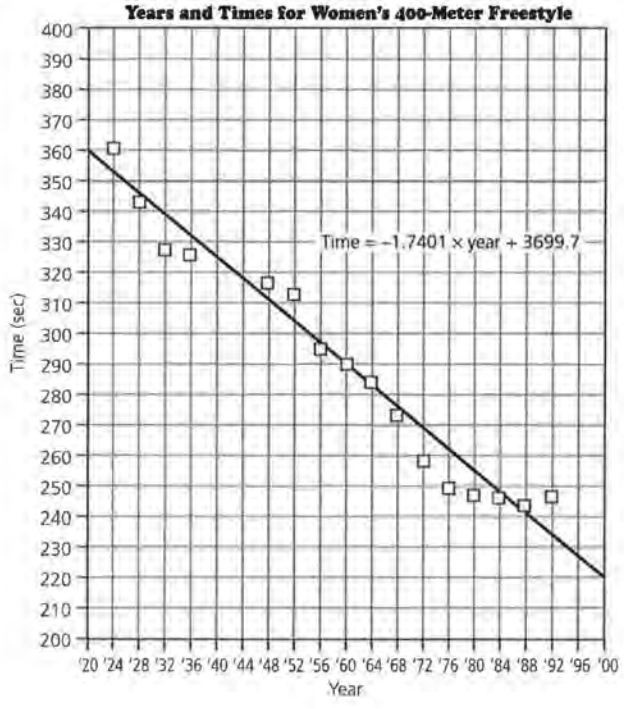
b. Draw lines that pass through the centroid $(\bar{x}, \bar{y})$ having varying slopes. Determine the sum of the squared residuals of these lines. The slope that creates the least sum of squared residuals is the slope of the least-squares regression line.

7. The advantage in using the least-squares regression line is that you know it will create the line having the least sum of the squared residuals and will be the best model, provided there are not some very great outliers.

The time data must be converted to one unit, either minutes or seconds. The graph uses seconds.

Year	Time (min:sec)	Time (seconds)	Time (minutes)
1924	6:02.2	362.2	6.04
1928	5:42.8	342.8	5.71
1932	5:28.5	328.5	5.48
1936	5.26.4	326.4	5.44
1948	5:17.8	317.8	5.30
1952	5:12.1	312.1	5.20
1956	4:54.6	294.6	4.91
1960	4:50.6	290.6	4.84
1964	4:43.3	283.3	4.72
1968	4:31.3	271.3	4.52
1972	4:19.0	259.0	4.32
1976	4:09.9	249.9	4.17
1980	4:08.8	248.8	4.15
1984	4:07.1	247.1	4.12
1988	4:03.9	243.9	4.07
1992	4:07.2	247.2	4.12

1.–2.



3. Every year the time for swimming the 400-meter freestyle decreases by 1.74 seconds.

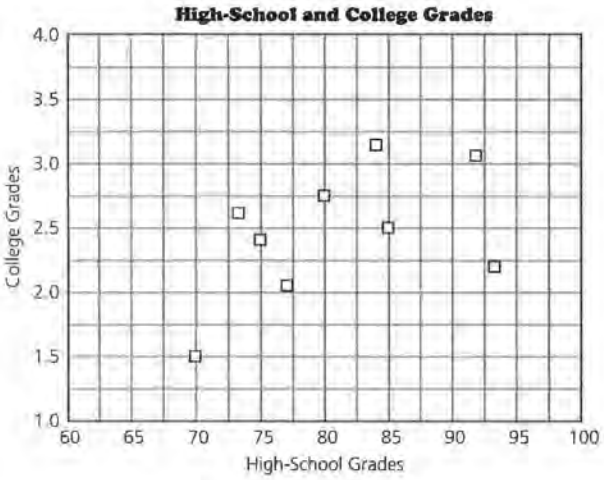
4. This equation makes the sum of the squared residuals a minimum.

5. $y = -1.74(1996) + 3699.7 = 226.7$ seconds, or 3 min, 46.7 sec; this prediction might be fairly accurate, because the year is so close to the rest of the data. However, this equation might not be very reliable to predict future times.

6. $y = -1.74(1940) + 3699.7 = 324.1$ seconds, or 5 min, 24 sec

7. There is a strong negative correlation between the year and the winning time for the women's 400-meter freestyle for the Olympic games since 1924.

1.



2. **a.** The correlation is 0.5134.

b. $r^2 = 0.264$; this number tells that approximately 26% of the variability in the college grades can be accounted for by knowing the high-school grade and using the linear-regression line.

c. Some other factors are being away from home and on your own, studying more or less, having a job, taking harder courses, being more dedicated, and so on.

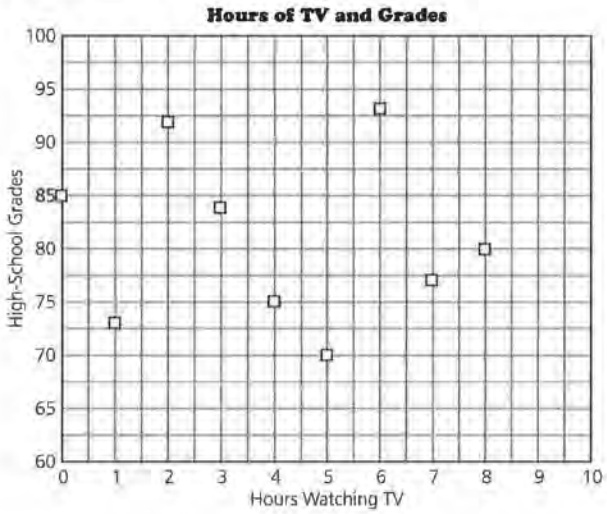
3. **a.** The least-squares regression line is college grade = 0.03 times high-school grade – 0.18.

b. The slope, 0.03, tells you that for every increase of one point in your high-school grade, your college grade will rise 0.03 of a point; or for every increase of 100 points in your high-school grade, your college grade will rise 3 points.

c. The prediction will be a college grade of $2.28 \approx 2.3$; college grade = 0.03(82) – 0.18 = 2.28

d. The high-school grade was 81; 0.03(81) – 0.18 = 2.25

4. **a.** Answers will vary; sample:



b. It would be nearly impossible to predict. Your best bet is to guess.

c. A negative correlation means that the association was decreasing; that is, as one increased the other decreased. The numerical value of the correlation coefficient r indicates whether that linear relationship between the variables were strong or weak. A correlation coefficient of zero means there did not exist any linear relationship between the two variables.

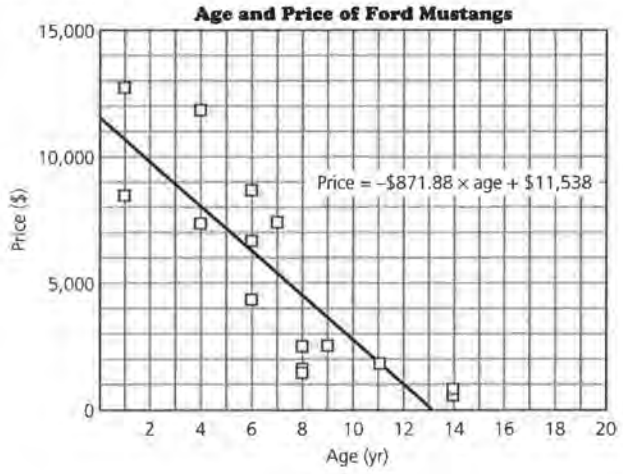
5. Research is probably referring to a high correlation between the grades in Latin and the SAT scores. In our study, we have found that there may be a high correlation coefficient and no cause-and-effect relationship. The conclusion therefore should be that the statement is very dangerous and should be taken with a "grain of salt."

1. a. From the graph, the value appears to be about 72 heartbeats per minute.

b. It appears that for the first ten years there is a negative association, after which it appears to level out at 72 heartbeats per minute.

c. The typical number of heartbeats per minute after age 20 appears to be 72 heartbeats per minute. This can be seen from the plot, because it appears to become a horizontal line $y = 72$.

2. a.



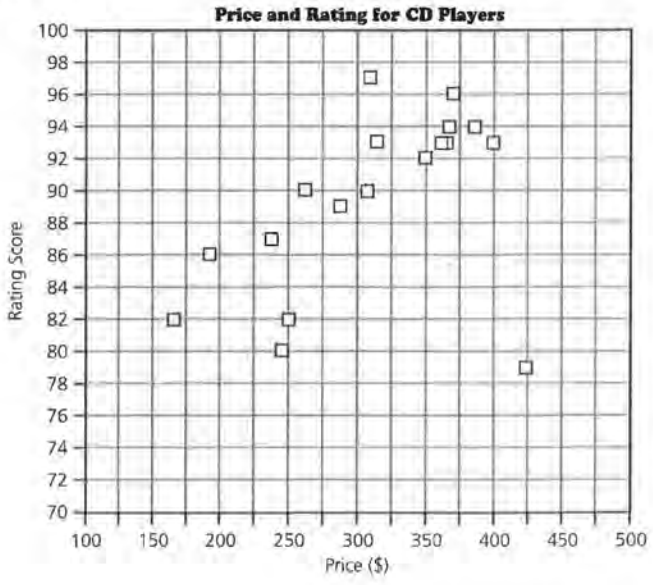
b. The slope, –871.88, indicates that for every increase of one year since 1993 in the age of a Ford Mustang the price of that automobile will decrease $871.88. The x-intercept indicates that at the age of 13.2 years (in the year 2006), the price of a Ford Mustang purchased in 1993 will be $0.00. The y-intercept indicates that a new Ford Mustang would have been priced at $11,538 in 1992.

c.

Year	Age (years)	Price ($)	Predicted Price ($)	Residuals
1980	14	900	–668.1	1568.1
1980	14	575	–668.1	1243.1
1983	11	1,950	1,947.5	2465.4
1985	9	2,695	3,691.3	–996.3
1986	8	1,795	4,563.2	–2768
1986	8	1,695	4,563.2	–2868
1986	8	2,500	4,563.2	–2063
1987	7	7,495	5,435	2060
1988	6	4,300	6,306.9	–2008
1988	6	6,800	6,306.9	493.08
1988	6	8,495	6,306.9	2188.1
1990	4	7,295	8,050.7	–755.7
1990	4	11,995	8,050.7	3944.3
1993	1	8,493	10,666	–2173
1993	1	12,799	10,666	2132.7

The sum of the squared residuals is 63,844,949.8. The least sum of squared residuals is that created using the least-squares regression line, because that is the basis on which that line is created. If the students calculate Toni's sum of squared residuals, they should get 93,599,750, which is 30 million larger.

3. a.



The plot shows a positive association, but the association does not appear to be very strong.

b. The correlation coefficient is 0.4798, which indicates that the association is not very strong.

c. The elimination of the apparent outlier (79, 425) would make the correlation coefficient higher.

d. The value of r^2 is 0.2302, which indicates that 23% of the rating can be predicted by knowing the price. Other factors contributing to a higher rating could be the quality of the sound reproduction, the size of the player, and so on.

e. The equation of the least-squares regression line is rating score = 0.0362572 × price + 78.15637.

f. The mean of the ratings is 89.44444, and the mean of the prices is 311.33333. Substituting the mean of the prices into the least-squares regression equation yields 0.0362572 (311.33333) + 78.15637 = 89.44444, which is the mean of the ratings scores.

4. Sample answers are given.

a. The correlation coefficient of zero indicates that there is no linear association between the two variables. A change in the value of one will not have any effect on the value of the other. The author has confused correlation with cause and effect.

b. The fact that there is a strong correlation means that the graph can be modeled using a linear equation. Again, the author of the article has confused cause and effect with correlation. There are confounding variables that are involved in this relationship.

5. Sample answers are given.

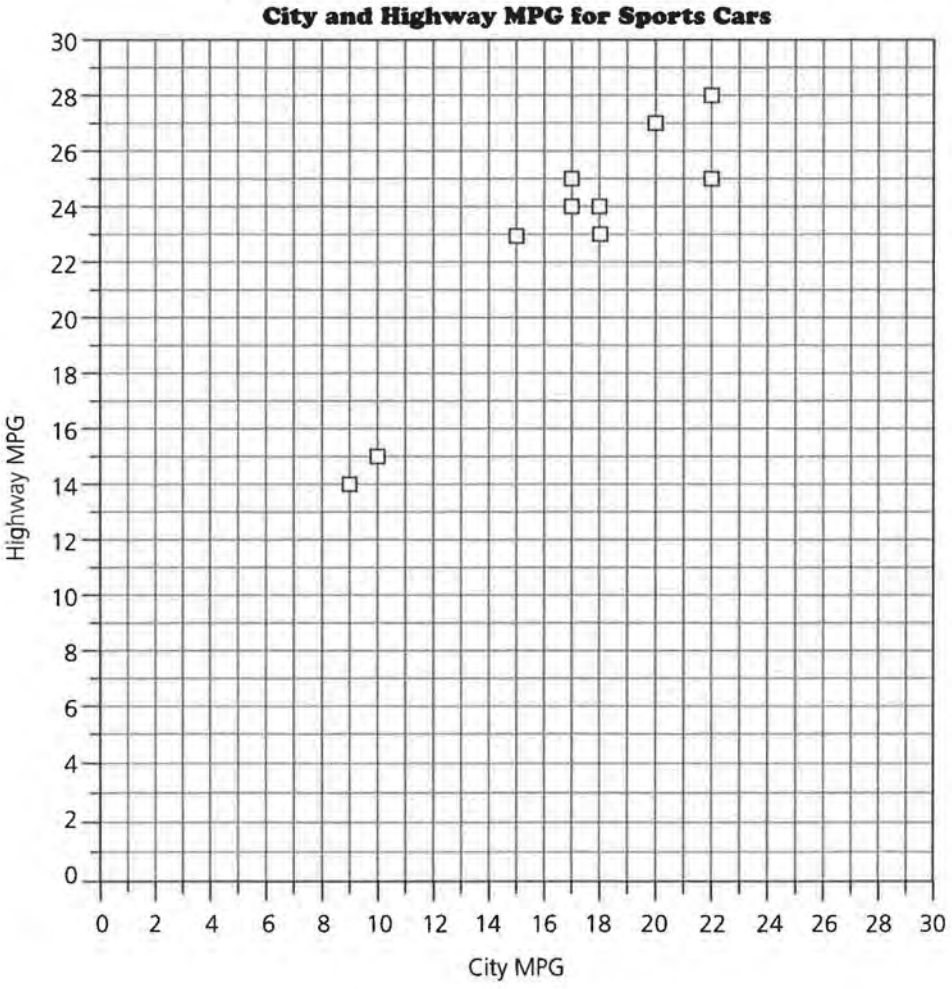
a. A residual is the value of the difference between the observed value of the data for a given x-value and the predicted value of the data at the same x-value. In statistics, the residuals are used to determine the *best* line to model a data set. The line that minimized the sum of the squares of the residuals is the best line for the model and is the least-squares regression line.

b. Correlation is the measure of the strength of the linear relationship between the two variables. Correlation is important because it helps determine whether it is reasonable to model the association between two variables with a linear equation.

c. The least-squares regression line is the line that passes through the centroid of the data and minimizes the sum of the squared residuals. It is important because it provides the *best* linear model when the data are linearly associated.

Lesson 1, Problem 5

NAME _____



Lesson 1, Problems 8 and 10

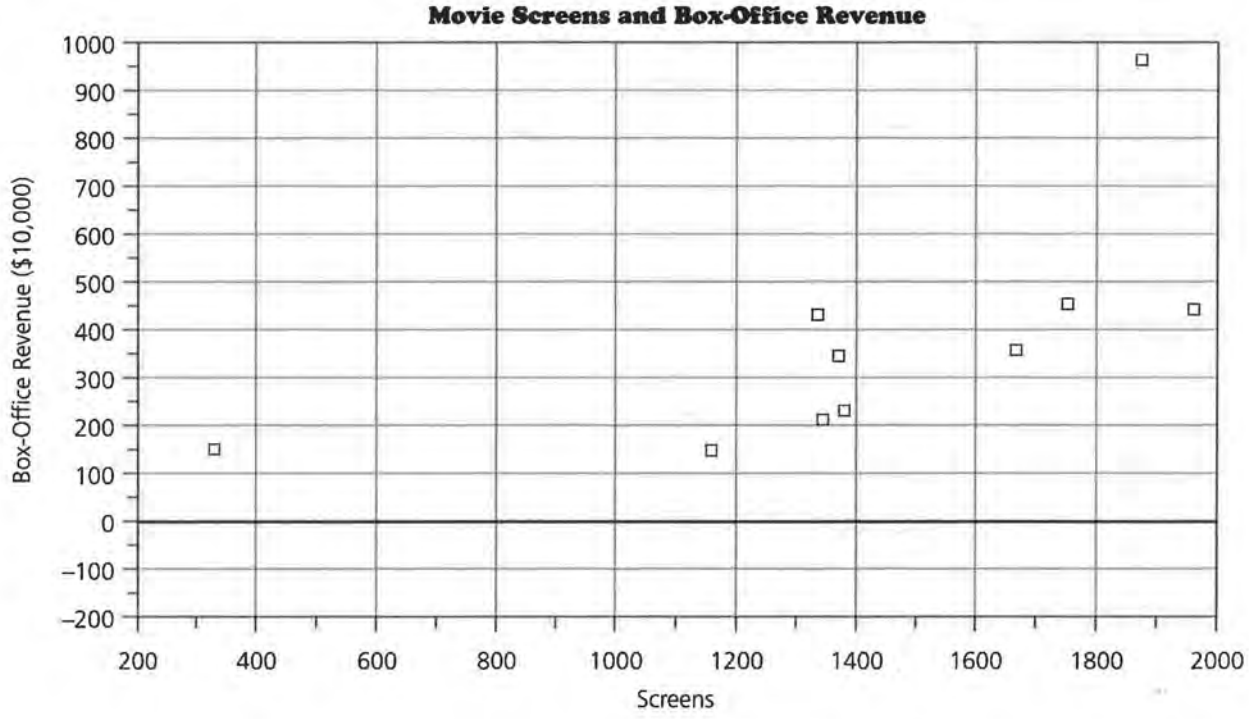
NAME _____

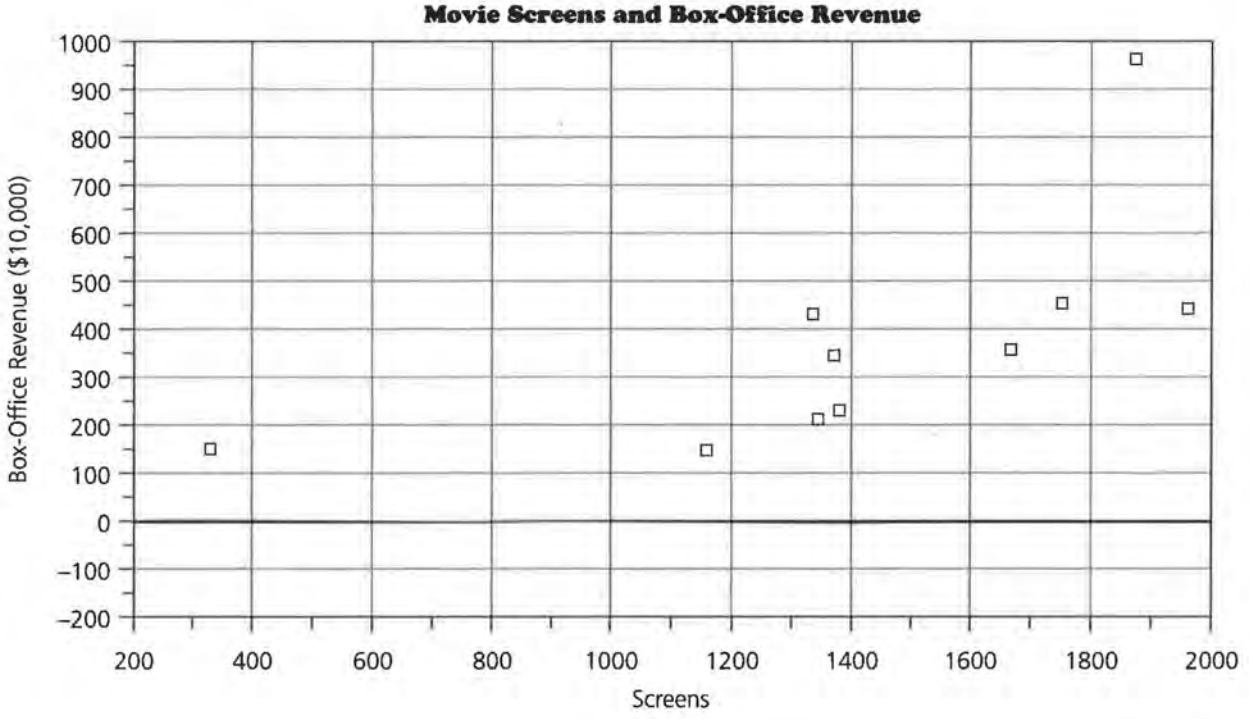
(City MPG, Hwy. MPG)	Predicted Hwy. MPG	Residual
(18, 24)	_____	_____
(22, 25)	_____	_____
(17, 25)	_____	_____
(10, 15)	_____	_____
(17, 24)	_____	_____
(9, 14)	_____	_____
(15, 23)	_____	_____
(22, 28)	_____	_____
(18, 23)	_____	_____
(17, 25)	_____	_____
(20, 27)	_____	_____

(City MPG, Hwy. MPG)	Predicted Hwy. MPG	Residual
(18, 24)	_____	_____
(22, 25)	_____	_____
(17, 25)	_____	_____
(10, 15)	_____	_____
(17, 24)	_____	_____
(9, 14)	_____	_____
(15, 23)	_____	_____
(22, 28)	_____	_____
(18, 23)	_____	_____
(17, 25)	_____	_____
(20, 27)	_____	_____

Lesson 2, Problems 2, 8, and 10
Lesson 4, Problem 1
Lesson 5, Problems 1 and 2
Lesson 6, Problem 2

NAME _____

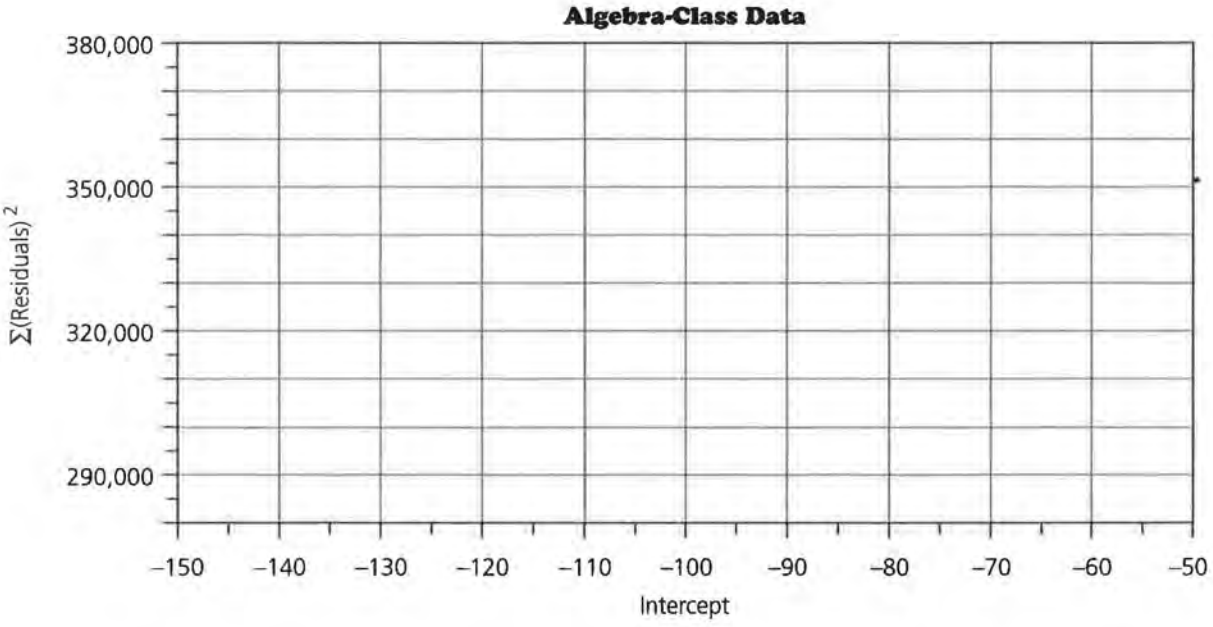




Lesson 5, Problems 6 and 7

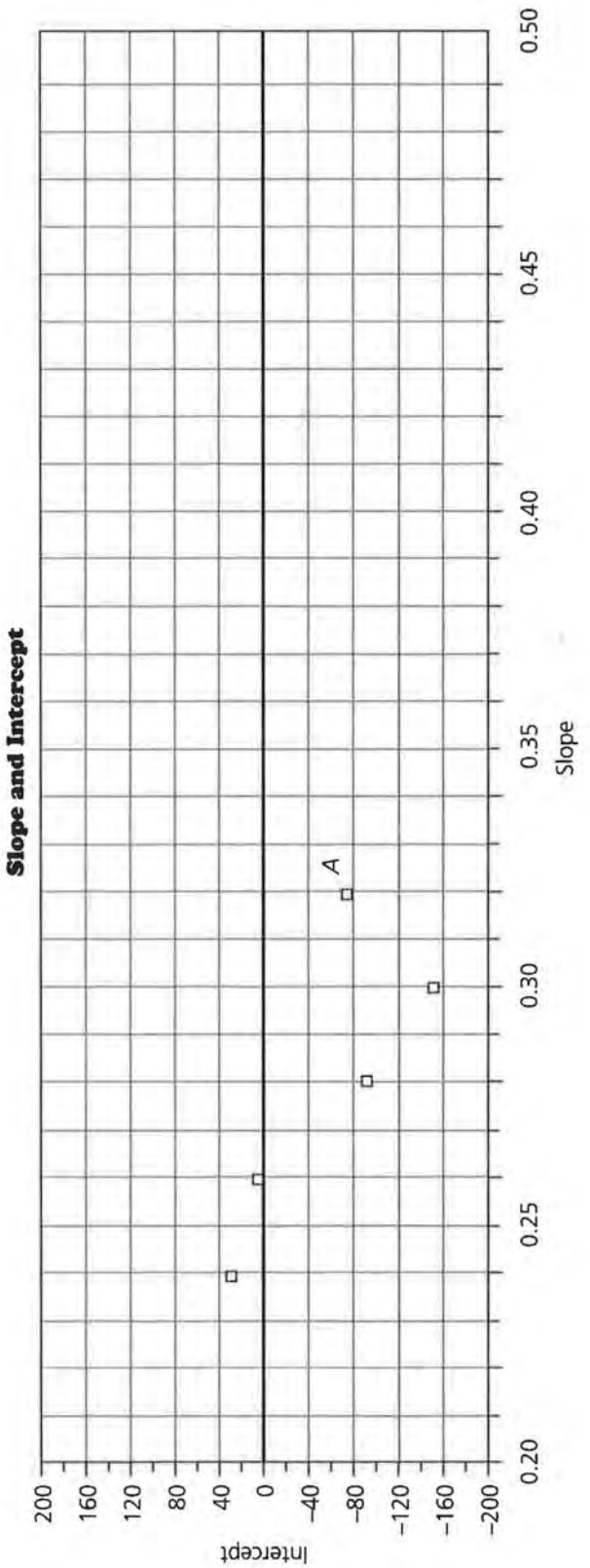
NAME _____

Slope	Point	Intercept	Sum of Squared Residuals
0.33	(1679, 352)	_____	_____
0.33	_____	_____	_____
0.33	_____	_____	_____
0.33	_____	_____	_____
0.33	_____	_____	_____



Lesson 6, Problem 1

NAME _____



Lesson 7, Problem 10

NAME _____

Equation	*a*	*b*	*c*	*x*-Intercepts	Minimum/ Maximum
$y = -2x^2$					
$y = 4x^2$					
$y = x^2 - 10x + 16$					
$y = x^2 - 10x - 11$					
$y = 3x^2 + 13x + 4$					
$y = x^2 - x - 2$					
$y = 8x^2 - 18x + 7$					
$y = x^2 - x + 2$					
$y = 2x^2 - x - 1$					
$y = x^2 - 2x + 1$					
$y = 3x^2 - 2x - 1$					
$y = x^2 - 4$					
$y = 9 - x^2$					

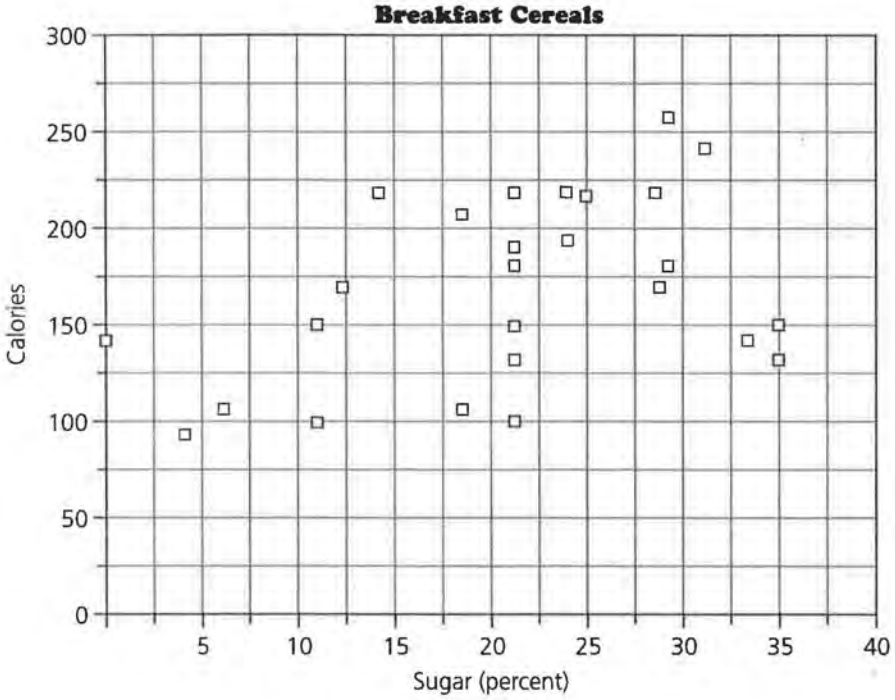
Lesson 8, Problems 3 and 5

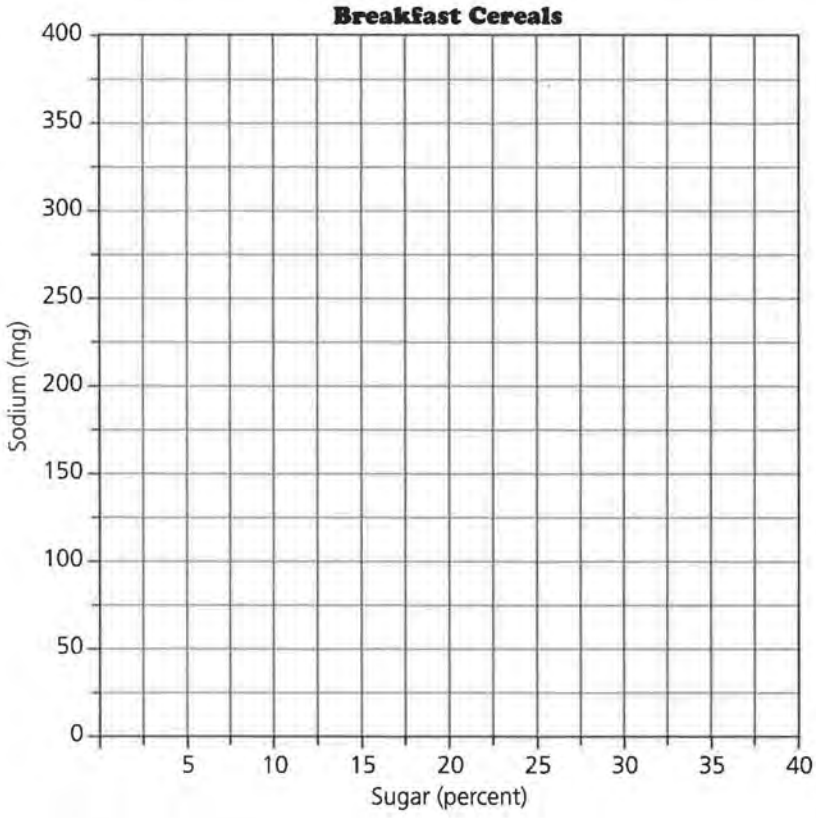
NAME _____

Movie	Screens	Income ($)	Predicted Income ($) $s(x - 1418.2) + 375.1$	Squared Residual	Quadratic-Error Expression
Wayne's World	1878	964	$s(1878 - 1418.2) + 375.1$	$[964 - (459.8s + 375.1)]^2$	$346{,}803.21 - 541{,}552.44s + 211{,}416.04s^2$
Memoirs of an Invisible Man	1753	460	_____	_____	_____
Stop or My Mom Will Shoot	1963	448	_____	_____	_____
Fried Green Tomatoes	1329	436	_____	_____	_____
Medicine Man	1363	353	_____	_____	_____
The Hand That Rocks the Cradle	1679	352	_____	_____	_____
Final Analysis	1383	230	_____	_____	_____
Beauty and the Beast	1346	212	_____	_____	_____
Mississippi Burning	325	150	_____	_____	_____
The Prince of Tides	1163	146	_____	_____	_____

Lesson 10, Problems 14 and 15

NAME _____





Data-Driven Mathematics
Procedures for Using the TI-83

I. Clear menus

ENTER will execute any command or selection. Before beginning a new problem, previous instructions or data should be cleared. Press **ENTER** after each step below.

1. To clear the function menu, **Y=**, place the cursor anyplace in each expression, **CLEAR**
2. To clear the list menu, **2nd MEM ClrAllLists**
3. To clear the draw menu, **2nd DRAW ClrDraw**
4. To turn off any statistics plots, **2nd STATPLOT PlotsOff**
5. To remove user-created lists from the Editor, **STAT SetUpEditor**

II. Basic information

1. A rule is active if there is a dark rectangle over the option. See Figure 1.

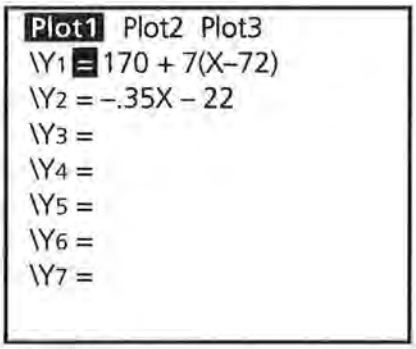


FIGURE 1

On the screen above, Y1 and Plot1 are active; Y2 is not. You may toggle Y1 or Y2 from active to inactive by putting the cursor over the = and pressing **ENTER**. Arrow up to **Plot1** and press **ENTER** to turn it off; arrow right to **Plot2** and press **ENTER** to turn it on, etc.

2. The Home Screen (Figure 2) is available when the blinking cursor is on the left as in the diagram below. There may be other writing on the screen. To get to the Home Screen, press **2nd QUIT**. You may also clear the screen completely by pressing **CLEAR**.

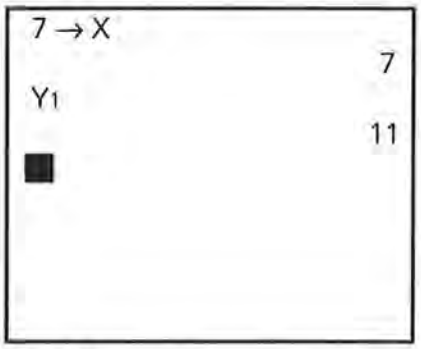


FIGURE 2

3. The variable x is accessed by the **X, T, Θ, *n*** key.
4. Replay option: **2nd ENTER** allows you to back up to an earlier command. Repeated use of **2nd ENTER** continues to replay earlier commands.
5. Under **MATH**, the **MATH** menu has options for fractions to decimals and decimals to fractions, for taking nth roots, and for other mathematical operations. **NUM** contains the absolute-value function as well as other numerical operations. (Figure 3)

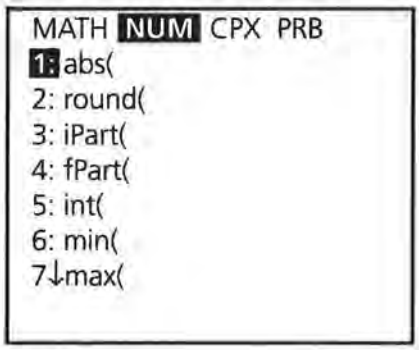


FIGURE 3

III. The STAT Menus

1. There are three basic menus under the **STAT** key: **EDIT**, **CALC**, and **TESTS**. Data are entered and modified in the **EDIT** mode; all numerical calculations are made in the **CALC** mode; statistical tests are run in the **TEST** mode.
2. **Lists and Data Entry**
   Data is entered and stored in Lists (Figure 4). Data will remain in a list until the list is cleared. Data can be cleared using **CLEAR** L_i or (List name), or by placing the cursor over the List heading and selecting **CLEAR ENTER**. To enter data, select **STAT EDIT** and with the arrow keys move the cursor to the list you want to use.

Type in a numerical value and press **ENTER.** Note that the bottom of the screen indicates the List you are in and the list element you have highlighted. 275 is the first entry in L1. (It is sometimes easier to enter a complete list before beginning another.)

L1	L2	L3
275	67	190
5311	144	120
114	64	238
2838	111	153
15	90	179
332	68	207
3828	94	153
L1 (1) = 275		

FIGURE 4

For data with varying frequencies, one list can be used for the data, and a second for the frequency of the data. In Figure 5 below, the L5(7) can be used to indicate that the seventh element in list 5 is 4, and that 25 is a value that occurs 4 times.

L4	L5	L6
55	1	-------
50	3	
45	6	
40	14	
35	12	
30	9	
25	4	
L5 (7) = 4		

FIGURE 5

3. **Naming Lists**
Six lists are supplied to begin with. L1, L2, L3, L4, L5, and L6 can be accessed also as **2nd** L_i. Other lists can be named using words as follows. Put the cursor at the top of one of the lists. Press **2nd INS** and the screen will look like that in Figure 6.

	L1	L2 1
	-------	-------
Name =		

FIGURE 6

The alpha key is on, so type in the name (up to five characters) and press **ENTER.** (Figure 7)

PRICE	L1	L2 2
	-------	-------
PRICE(1) =		

FIGURE 7

Then enter the data as before. (If you do not press **ENTER**, the cursor will remain at the top and the screen will say "error: data type.") The newly named list and the data will remain until you go to Memory and delete the list from the memory. To access the list for later use, press **2nd LIST** and use the arrow key to locate the list you want under the **NAMES** menu. You can accelerate the process by typing **ALPHA P** (for price). (Figure 8) Remember, to delete all but the standard set of lists from the editor, use SetUp Editor from the **STAT** menu.

NAMES OPS MATH
↑PRICE
: RATIO
: RECT
: RED
: RESID
: SATM
↓SATV

FIGURE 8

4. Graphing Statistical Data

General Comments

- Any graphing uses the **GRAPH** key.
- Any function entered in Y1 will be graphed if it is active. The graph will be visible only if a suitable viewing window is selected.
- The appropriate x- and y-scales can be selected in **WINDOW**. Be sure to select a scale that is suitable to the range of the variables.

Statistical Graphs

To make a statistical plot, select **2nd Y=** for the **STAT PLOT** option. It is possible to make three plots concurrently if the viewing windows are identical. In Figure 9, Plots 2 and 3 are off, Plot 1 is a scatter plot of the data (Costs, Seats), Plot 2 is a scatter plot of (L3, L4), and Plot 3 is a box plot of the data in L3.

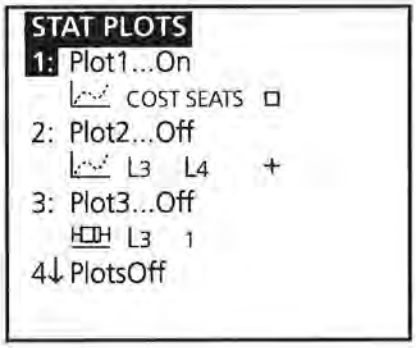


FIGURE 9

Activate one of the plots by selecting that plot and pressing **ENTER**.

Choose **ON**, then use the arrow keys to select the type of plot (scatter, line, histogram, box plot with outliers, box plot, or normal probability plot). (In a line plot, the points are connected by segments in the order in which they are entered. It is best used with data over time.) Choose the lists you wish to use for the plot. In the window below, a scatter plot has been selected with the x-coordinate data from COSTS, and the y-coordinate data from SEATS. (Figure 10) (When pasting in list names, press **2nd LIST**, press **ENTER** to activate the name, and press **ENTER** again to locate the name in that position.)

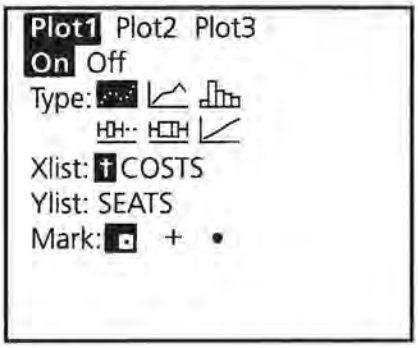


FIGURE 10

For a histogram or box plot, you will need to select the list containing the data and indicate whether you used another list for the frequency or are using 1 for the frequency of each value. The x-scale selected under **WINDOW** determines the width of the bars in the histogram. It is important to specify a scale that makes sense with the data being plotted.

5. Statistical Calculations

One-variable calculations such as mean, median, maximum value of the list, standard deviation, and quartiles can be found by selecting **STAT CALC 1-Var Stats** followed by the list in which you are interested. Use the arrow to continue reading the statistics. (Figures 11, 12, 13)

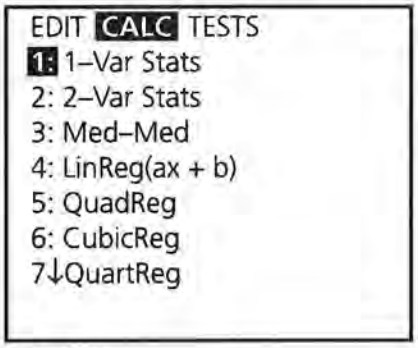


FIGURE 11

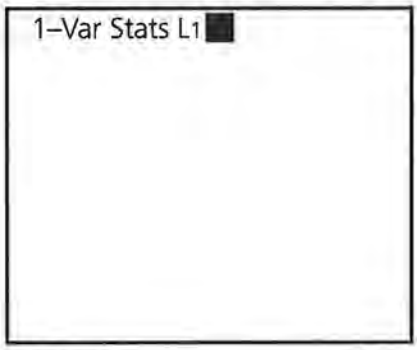


FIGURE 12

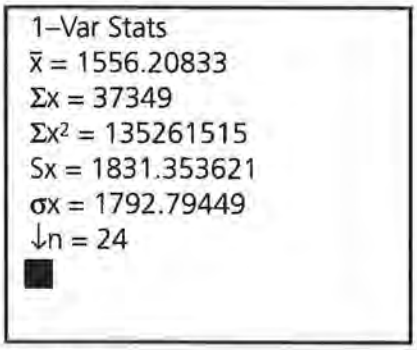


FIGURE 13

Calculations of numerical statistics for bivariate data can be made by selecting two-variable statistics. Specific lists must be selected after choosing the **2-Var Stats** option. (Figure 14)

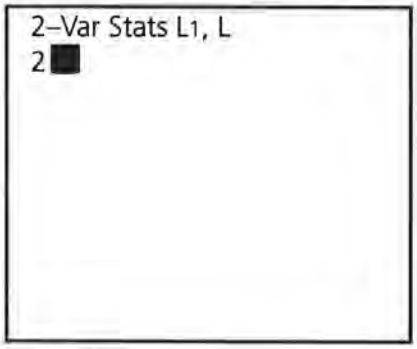


FIGURE 14

Individual statistics for one- or two-data sets can be obtained by selecting **VARS Statistics,** but you must first have calculated either 1-Var or 2-Var Statistics. (Figure 15)

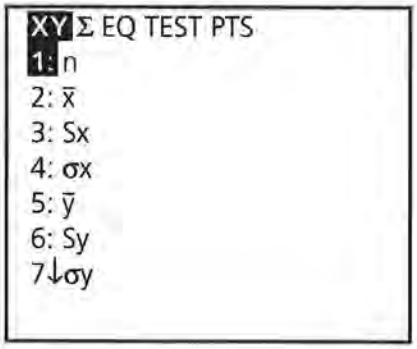


FIGURE 15

6. **Fitting Lines and Drawing Their Graphs**
Calculations for fitting lines can be made by selecting the appropriate model under **STAT CALC: Med-Med** gives the median fit regression, **LinReg** the least-squares linear regression, and so on. (Note the only difference between **LinReg (ax+b)** and **LinReg (a+bx)** is the assignment of the letters a and b.) Be sure to specify the appropriate lists for x and y. (Figure 16)

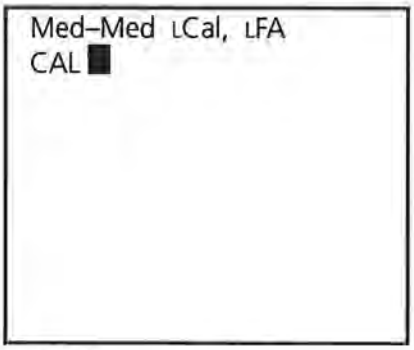


FIGURE 16

To graph a regression line on a scatter plot, follow the steps below:

- Enter your data into the Lists.
- Select an appropriate viewing window and set up the plot of the data as above.
- Select a regression line followed by the lists for x and y, **VARS Y-VARS Function** (Figures 17, 18) and the Y_i you want to use for the equation, followed by **ENTER.**

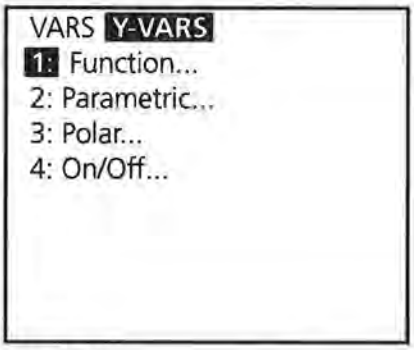


FIGURE 17

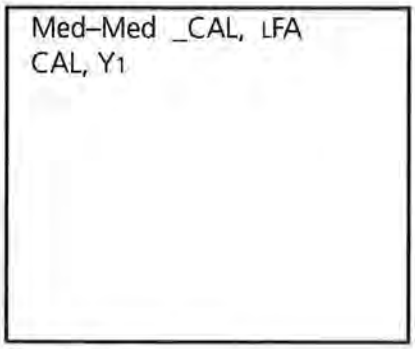


FIGURE 18

The result will be the regression equation pasted into the function Y1. Press **GRAPH** and both the scatter plot and the regression line will appear in the viewing window. (Figures 19, 20)

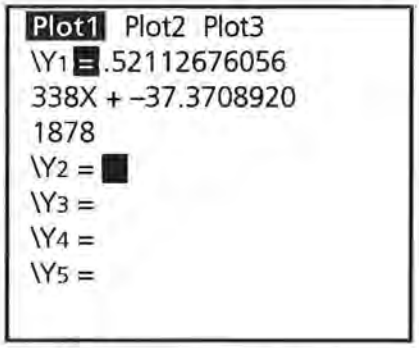


FIGURE 19

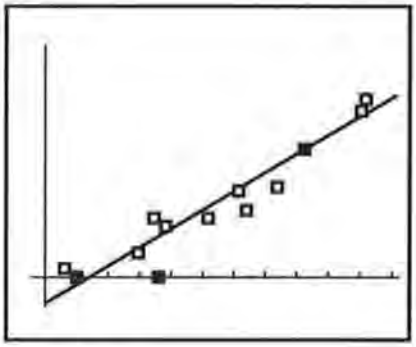


FIGURE 20

- There are two cursors that can be used in the graphing screen.

TRACE activates a cursor that moves along either the data (Figure 21) or the function entered in the Y-variable menu (Figure 22). The coordinates of the point located by the cursor are given at the bottom of the screen.

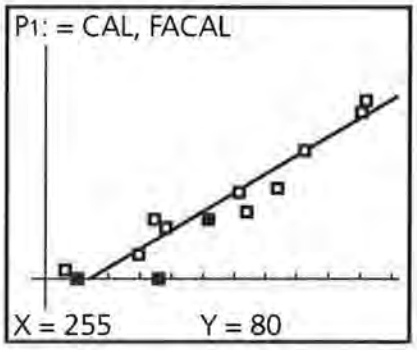


FIGURE 21

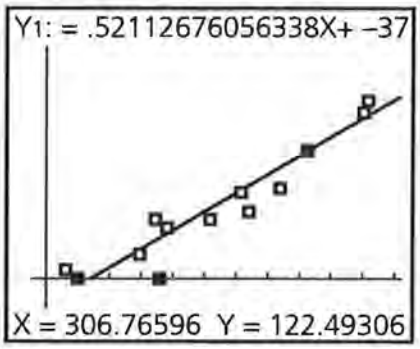


FIGURE 22

Pressing **GRAPH** returns the screen to the original plot. The up arrow key activates a cross cursor that can be moved freely about the screen using the arrow keys. See Figure 23.

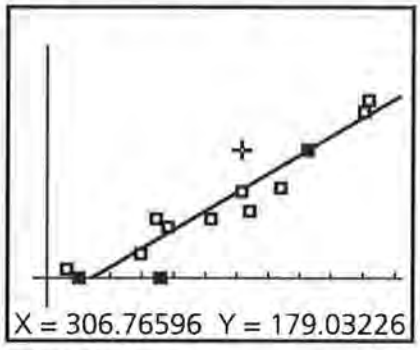


FIGURE 23

Exact values can be obtained from the plot by selecting **2nd CALC** Value. Select **2nd CALC Value ENTER**. Type in the value of x you would like to use, and the exact ordered pair will appear on the screen with the cursor located at that point on the line. (Figure 24)

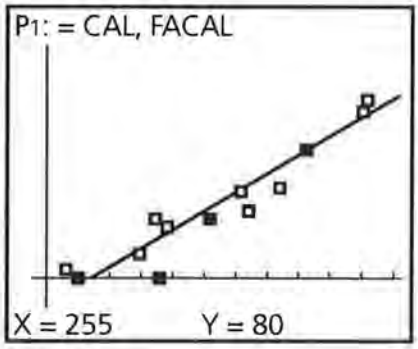


FIGURE 24

IV. Evaluating an expression

To evaluate $y = .225x - 15.6$ for $x = 17$, 20, and 24, you can:

1. Type the expression in Y1, return to the home screen, 17 **STO X,T,Θ,n ENTER, VARS Y-VARS Function Y1 ENTER ENTER.** (Figure 25)

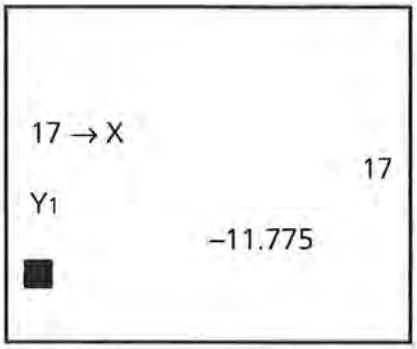


FIGURE 25

Repeat the process for $x = 20$ and 24.

2. Type $17^2 - 4$ for $x = 17$, **ENTER** (Figure 26). Then use **2nd ENTRY** to return to the arithmetic line. Use the arrows to return to the value 17 and type over to enter 20.

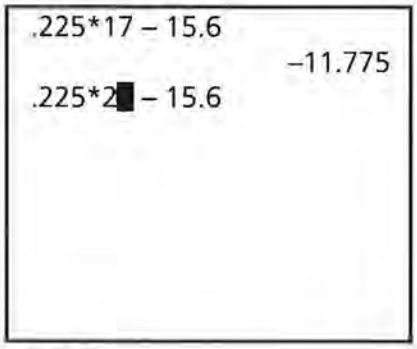


FIGURE 26

You can also find the value of x by using the table command. Select **2nd TblSet** (Figure 27). (Y1 must be turned on.) Let **TblStart** = 17, and the increment **ΔTbl** = 1.

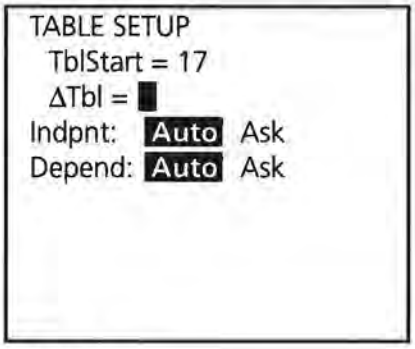


FIGURE 27

Select **2nd TABLE** and the values of x and y generated by the equation in Y1 will be displayed. (Figure 28)

X	Y1	
17	-11.78	
18	-11.55	
19	-11.33	
20	-11.1	
21	-10.88	
22	-10.65	
23	-10.43	
X = 17		

FIGURE 28

V. Operating with Lists

1. A list can be treated as a function and defined by placing the cursor on the label above the list entries. List 2 can be defined as L1 + 5. (Figure 29)

L1	L2	L3
275	-------	190
5311		120
114		238
2838		153
15		179
332		207
3828		153
L2 = L1 + 5		

FIGURE 29

Pressing **ENTER** will fill List 2 with the values defined by L1+5. (Figure 30)

L1	L2	L3
275	280	190
5311	5316	120
114	119	238
2838	2843	153
15	20	179
332	337	207
3828	3833	153
L2(1) = 280		

FIGURE 30

2. List entries can be cleared by putting the cursor on the heading above the list, and selecting **CLEAR** and **ENTER.**

3. A list can be generated by an equation from Y= over a domain specified by the values in L1 by putting the cursor on the heading above the list entries. Select **VARS Y-VARS Function Y1 ENTER (L1) ENTER.** (Figure 31)

L1	L2	L3
120	12	-------
110	14	
110	12	
110	11	
100	?	
100	6	
120	9	
L3 = Y1(L1)		

FIGURE 31

4. The rule for generating a list can be attached to the list and retrieved by using quotation marks (**ALPHA** +) around the rule. (Figure 32) Any change in the rule (Y1 in the illustration) will result in a change in the values for L1. To delete the rule, put the cursor on the heading at the top of the list, press **ENTER,** and then use the delete key. (Because L1 is defined in terms of **CAL,** if you delete **CAL** without deleting the rule for L1 you will cause an error.)

CAL	FACAL	L1 5
255	80	-------
305	120	
410	180	
510	250	
320	90	
370	125	
500	235	
L1 = "Y1(LCAL)"		

FIGURE 32

VI. Using the DRAW Command

To draw line segments, start from the graph of a plot, press **2ND DRAW,** and select **Line(.** (Figure 33)

DRAW POINTS STO
1: ClrDraw
2: Line(
3: Horizontal
4: Vertical
5: Tangent(
6: DrawF
7↓Shade(

FIGURE 33

This will activate a cursor that can be used to mark the beginning and ending of a line segment. Move the cursor to the beginning point and press **ENTER;** use the cursor to mark the end of the segment, and press **ENTER** again. To draw a second segment, repeat the process. (Figure 34)

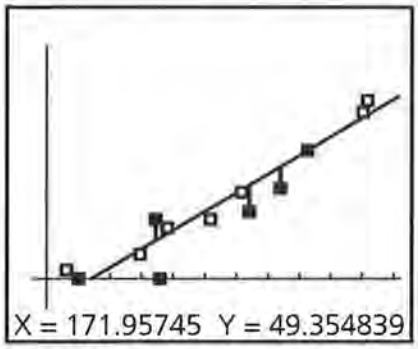


FIGURE 34

VII. Random Numbers

To generate random numbers, press **MATH** and **PRB**. This will give you access to a random number function, **rand**, that will generate random numbers between 0 and 1 or **randInt(** that will generate random numbers from a beginning integer to an ending integer for a specified set of numbers. (Figure 35) In Figure 36, five random numbers from 1 to 6 were generated.

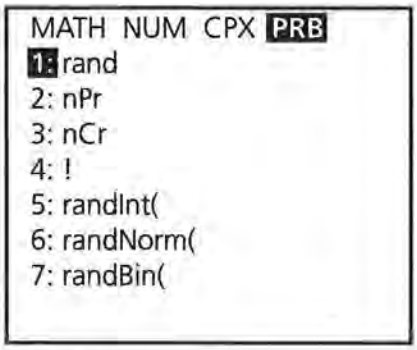


FIGURE 35

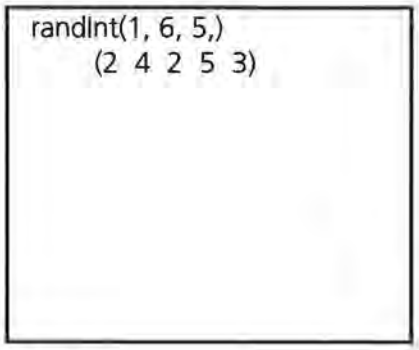


FIGURE 36

Pressing **ENTER** will generate a second set of random numbers.

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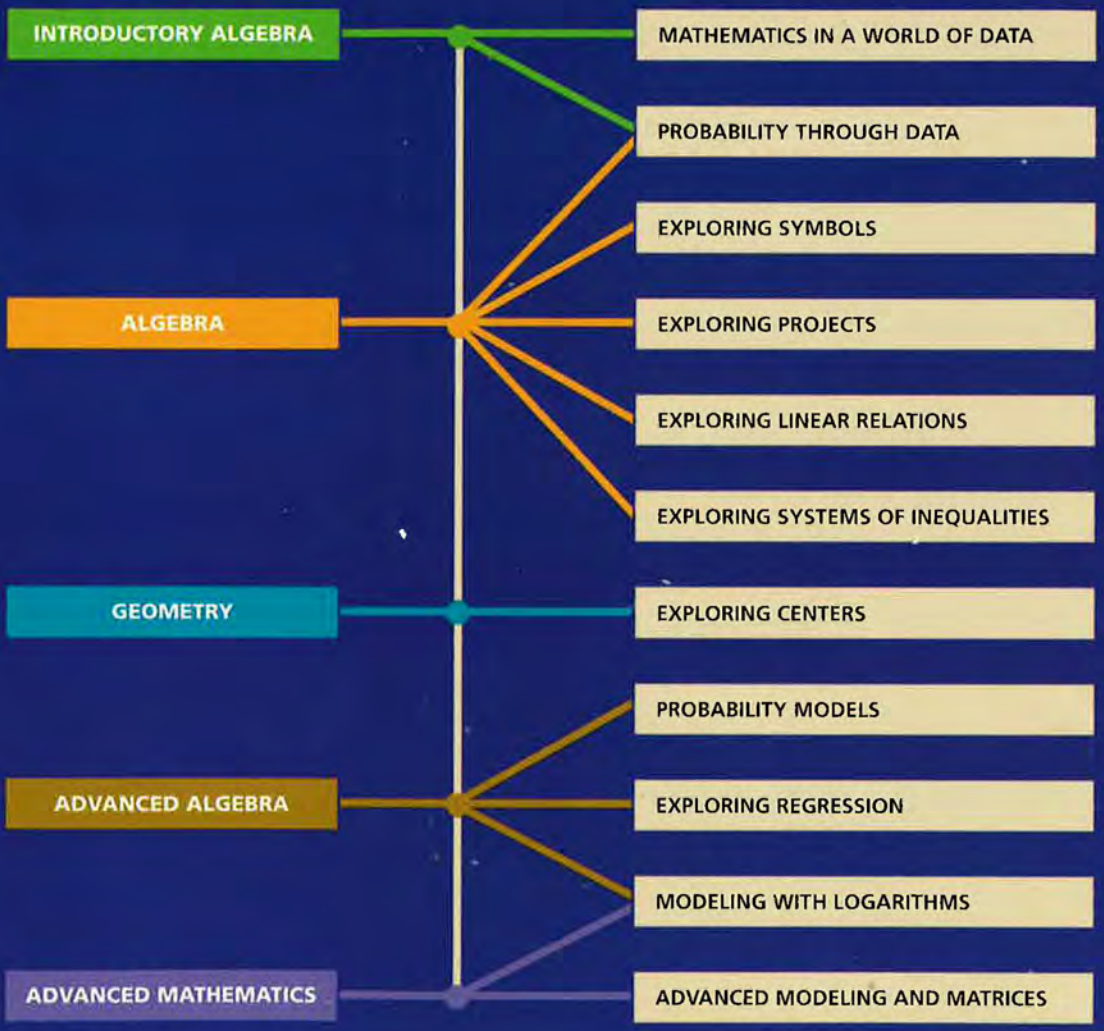

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