Exploring Geometric Probabilities with Buffon's Coin Problem

Kady Schneiter
Utah State University
kady.schneiter@usu.edu

Published: October 2012

Overview of Lesson
Investigate Buffon's coin problem using physical or virtual manipulatives (or both) to make connections between geometry and probability, to identify empirical and theoretical probabilities and to discuss the relationship between them.

GAISE Components
This investigation follows the four components of statistical problem solving put forth in the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report*. The four components are: formulate a question, design and implement a plan to collect data, analyze the data by measures and graphs, and interpret the results in the context of the original question. This is a GAISE Level B activity.

Common Core State Standards for Mathematical Practice
2. Reason abstractly and quantitatively.
5. Use appropriate tools strategically.

Common Core State Standards Grade Level Content (Grade 7)
7. G. 6. Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.
7. SP. 1. Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.

NCTM Principles and Standards for School Mathematics
Data Analysis and Probability Standards for Grades 6-8

Develop and evaluate inferences and predictions that are based on data:
- use conjectures to formulate new questions and plan new studies to answer them.

Understand and apply basic concepts of probability:
- use proportionality and a basic understanding of probability to make and test conjectures about the results of experiments and simulations.

Prerequisites

Students will know the area formulas for squares and rectangles.
Students will have a basic understanding of probability.
Students will be familiar with the ideas of proportion and simulation.
Students will understand the terms "event," "sample space," and "estimate."

Learning Targets

Students will define 'geometric probability', 'empirical probability', 'theoretical probability'.
Students will use simulation to identify an empirical solution to Buffon's coin problem.
Students will use area formulas to identify a theoretical solution to Buffon's coin problem.
Students will observe that as the number of trials increases, the empirical probability tends to approach the theoretical probability.

Time Required

Approximately 50 minutes.

Materials Required

- Coins and square grid (the length of the diameter of the coin should be less than the length of a square on the grid – possibilities are plastic lids on floor tiles, or pennies on graph paper). A blank grid for pennies is provided on page 9.
- Applet at www.math.usu.edu/~schneit/CTIS/BC.
- Pencil and paper for record keeping and note taking.
- Calculator.

Instructional Lesson Plan

This lesson plan involves two parts: an empirical investigation and a theoretical one. Each of these investigations follows the four GAISE components: (1) Formulate a Question, (2) Collect Data, (3) Analyze the Data, (4) Interpret the Data – answer the question.

Preparation: Divide students into small groups (2-4). Prepare a 'coin' and a grid of square tiles for each group (the length of the diameter of the coin should be less than the length of a side of a tile in the grid) but do not yet distribute these. Assign one member of each group to be the recorder.

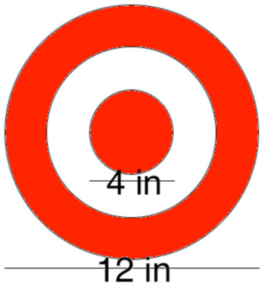
Background: Georges-Louis Leclerc, Comte de **Buffon** (1707 – 1788) was a French mathematician and naturalist. His 'coin problem' is an early exercise in geometric probability, a field in which probabilities are concerned with proportions of areas (lengths or volumes) of geometric objects under specified conditions.

Examples of questions that deal with geometric probabilities are:

- What is the probability of hitting the bull's eye when a dart is thrown randomly at a target?
- What is the probability that a six-color spinner lands on red?

Geometric probabilities can be estimated using empirical (experimental) methods or identified exactly (theoretical probability) using analytical methods.

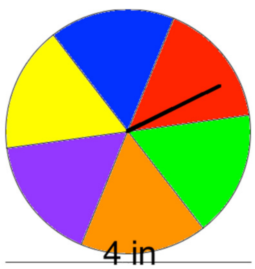
Definition: An empirical probability is the proportion of times an event of interest occurs in a set number of repetitions of an experiment.



Throw 50 darts at the target.

5 darts hit the bull's eye.

The empirical probability of hitting the bull's eye is 5/50 = 1/10.

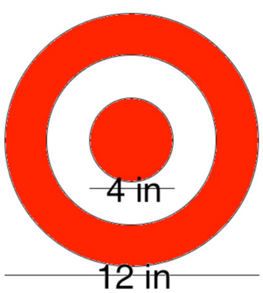


Spin the spinner 100 times.

Spinner lands on red 8 times.

Empirical probability = 8/100 = 2/25.

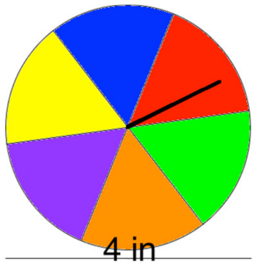
Definition: A theoretical probability is the proportion of times an event of interest would be expected to occur in an infinite number of repetitions of an experiment. For a geometric probability, this is the ratio of the area of interest (e.g. bull's eye) to the total area (e.g. target).



Area of target = $\pi * 6^2$

Area of bull's eye = $\pi * 2^2$

Theoretical probability of hitting bull's eye = $\frac{\pi * 2^2}{\pi * 6^2} = {}^1/_9$



Area of spinner = $\pi * 2^2$

Area of red section = $\frac{1}{6} \pi * 2^2$

Theoretical probability of landing on red = $\frac{\frac{1}{6}\pi * 2^2}{\pi * 2^2} = \frac{1}{6}$

Investigation (Empirical)

The GAISE Statistical Problem-Solving Procedure

I. Formulate Question(s)
Buffon's coin problem: What is the probability that a coin, tossed randomly at a grid, will land entirely within a tile rather than across the tile boundaries? (Again, for the purposes of this activity, assume that the diameter of the coin is less than the length of a side of the tile.)

II. Design and Implement a Plan to Collect the Data

- *Discuss as a class*: how would we identify an empirical solution to Buffon's coin problem? (This corresponds to prompt 1 on the task sheet.)
  (After students propose tossing coins at a grid, discuss details of the experiment: how many times will they throw the coin? How will the coin be tossed? Does skill matter? Will they count the times that the coin lands on a boundary or the times it lands entirely within a tile? Will each group do this the same way? What difference will it make if they do not? (For purposes of later discussion it will be helpful if everyone considers the event that the coin lands entirely within a tile.) Who will record the outcome of each toss? How will this count translate into an empirical probability?)

- *Experiment*: Instruct each group to conduct the experiment, as designed by the class. (Task sheet prompt 2.)

III. Analyze the Data
Instruct each group to use the data they gathered to compute an empirical probability of the event they considered.

IV. Interpret the Results

- *Discuss as a class*:
  - Summarize the probabilities on the board or other place where all students can see. Ask students what they observe about the empirical probabilities computed by the groups. (They are not all the same, most are similar, a few may differ by a lot, if the experiment were repeated different answers would be obtained).

- Is it possible to get a more stable answer? (Yes, repeat the experiment more times, combine data from different groups).
- Ask the students what they would expect to see if the coin could be tossed an infinite number of times. Why would they expect to see this? (Task sheet prompt 3.)

Investigation (Theoretical)

I. Formulate Question(s)

Recall Buffon's coin problem: What is the probability that the coin, tossed randomly at a grid, will land entirely within a tile rather than across the tile boundaries?

II. Design and Implement a Plan to Collect the Data

- *Discuss as a class*: how would we identify a theoretical solution to Buffon's coin problem? (Task sheet prompt 4.)
  (Just outline the process here: identify the shape of the region within the tile in which the coin must land to be entirely within the tile, look at the ratio of the area of that shape to the area of a tile. Students will work out the details with their groups in the next segment.)
- *Explore*: Again working in groups, ask students:
  - To formulate a conjecture about the relationship between theoretical and empirical probabilities. (Task sheet prompt 5.)
  - To identify the shape of the region within the tile in which the coin must land to be entirely within a tile. (This will be challenging for some students. The key is to consider where the center of the coin lands and how close the center can be to the edge of a tile while the coin is not on a boundary. The applet cited above can be used to direct students' thinking. With the radio button 'Show centers' selected, for repeated coin tosses, the applet will mark where the center of the coin lands. The color of the mark differs depending on whether the coin crosses tile boundaries or not, thus the region of interest is clearly visible.)

III. Analyze the Data

Instruct each group to use their observations from the physical experiment, the applet, and the group discussion to compute a theoretical probability that a coin tossed randomly at a grid lands entirely within a single tile. (Task sheet prompt 6.)

IV. Interpret the Results

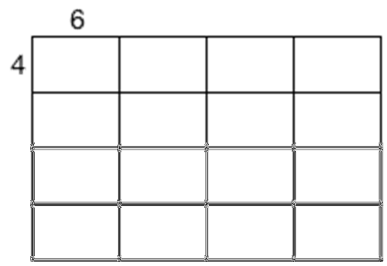
- *Discuss as a class*: Summarize students' solutions on the board. Discuss observations about these solutions. Bring the class to a consensus about the solution. Observe that there is only one solution and it will not vary with further investigation.

Synthesis

- *Discuss as a class*: What seems to be the relationship between the empirical and the theoretical probabilities? (As the number of trials increases, the empirical probability tends to converge to the theoretical one). (Task sheet prompt 7.)

Assessment
Students will identify the theoretical solution to Buffon's coin problem for a grid composed of non-square tiles. (The applet facilitates investigation with parallelograms and rectangles.)

Suppose you have a coin with diameter = 2 cm and a grid of rectangles with side lengths 4cm and 6cm as show in the figure below.



1. Describe how you would find the empirical probability that the coin would land completely within a tile when tossed at the grid.

2. Find the theoretical probability that the coin would land completely within a tile when tossed at the grid.

3. Describe the relationship between the theoretical and empirical probabilities.

nswers

1. Toss the coin at the grid repeatedly, record the number of times the coin lands entirely within a tile, divide this count by the total number of tosses to obtain the probability
2. 1/3
3. As the number of times the experiment is repeated increases, the empirical probability will approach the theoretical probability

Possible Extensions

Repeat the problem for grids with differently shaped tiles, for different sizes of tiles or coins, or ask students to suggest their own problems in geometric probability. The applet previously cited will allow students to experiment with differently shaped tiles, with tiles and coins of different problems, and with the chessboard problem described below.

The chessboard problem (*Plus Magazine*, 2004; Turner, 2006) is an interesting extension of Buffon's coin problem in which the probability of interest is that of landing completely on a corner of a tile – this also can be investigated using a variety of grid shapes.

References

1. Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M. & Scheaffer, R. (2007) Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report - A Pre-K-12 Curriculum Framework. American Statistical Association. Alexandria VA. www.amstat.org/education/gaise

2. *Plus Magazine* (2004). Puzzle page: High roller. http://plus.maths.org/content/os/issue31/puzzle/index .

3. Schneiter, K. (2011). Buffon's Coin Problem and Beyond. *Teaching Statistics*. **33**(2), 34-37.

4. Turner, P. (2006). Coin on a chessboard. *Australian Mathematics Teacher*, **62**(3), 12–16.

Exploring Geometric Probabilities with Buffon's Coin Problem Activity Sheet

Definitions:
1. Geometric probability: a probability concerned with proportions of areas (lengths or volumes) of geometric objects under specified conditions.

2. Empirical probability: the proportion of times an event of interest occurs in a set number of repetitions of an experiment.

3. Theoretical probability: the proportion of times an event of interest would be expected to occur in an infinite number of repetitions of an experiment.

Investigation:
Consider the question: What is the probability that a coin, tossed randomly at a grid, will land entirely within a tile rather than across tile boundaries?

1. How would we identify an *empirical* probability that "a coin, thrown randomly at a grid, will land entirely within a tile rather than across tile boundaries?"

2. Work with your group to compute an empirical probability that the coin lands within a tile. Record your observations below:

3. What would you expect to see if the coin could be tossed an infinite number of times at the grid? Why would you expect to see this?

4. How would we identify a *theoretical* probability that "a coin, thrown randomly at a grid, will land entirely within a tile rather than across tile boundaries?" How is this question different from question 1?

5. Formulate a conjecture: what is the relationship between the empirical and theoretical probabilities?

6. Work with you group to compute the theoretical probability that the coin lands within a tile. Record your work below:

7. Compare the empirical and theoretical probabilities you found. How do your results relate to the conjecture you proposed?

Example Grid
