

# Odd or Even? The Addition and Complement Principles of Probability



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## Overview of Lesson

After an exploration activity and a class discussion, students will be able to determine when events are mutually exclusive, and then use the Addition Principle to calculate the probability of mutually exclusive and non-mutually exclusive events. Furthermore, the students will explore the probabilities of complementary events and develop the Complement Principle.

## GAISE Components

This investigation follows the four components of statistical problem solving put forth in the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report*. The four components are: formulate a question, design and implement a plan to collect data, analyze the data, and interpret results in the context of the original question.

This is a **GAISE Level B** activity.

## Common Core State Standards for Mathematical Practice

MP1. Make sense of problems and persevere in solving them.

MP3. Construct viable arguments and critique the reasoning of others.

## Learning Objectives Alignment with Common Core State Standards and NCTM Principles and Standards for School Mathematics

Learning Objectives	Common Core	NCTM PSSM
Students will use simulations to explore empirical probabilities and compare them to theoretical probabilities.	<b>7.SP.C.8.C.</b> Design and use a simulation to generate frequencies for compound events.	<b>Grades 6-8</b> Understand and apply basic concepts of probability: use proportionality and a basic understanding of probability to make and test conjectures about the results of experiments and simulations; <b>Grades 9-12</b> Understand and apply basic concepts of probability: use simulations to

		construct empirical probability distributions;
Students will determine sample spaces using organized tables.	<b>7.SP.C.8.B.</b> Represent sample spaces for compound events using methods such as organized lists, tables, and tree diagrams.	<b>Grades 6-8</b> Understand and apply basic concepts of probability: compute probabilities for simple compound events, using such methods as organized lists, tree diagrams, and area models.
Students will use sample spaces to compute probabilities of compound events using unions and complements.	<b>S-CP.1.</b> Describe events as subsets of a sample space, using characteristics of the outcomes or as unions, intersections, or complements of other events.	<b>Grades 9-12</b> Understand and apply basic concepts of probability: understand the concepts of sample space and probability distribution and construct sample spaces and distributions in simple cases.
Students will discover and use the Addition Principle for mutually exclusive and non-mutually exclusive events: If A and B are mutually exclusive, then $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$ . If A and B are not mutually exclusive, then $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \text{ and } B)$ .	<b>S-CP.7.</b> Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ and interpret the answer in terms of the model.	<b>Grades 6-8</b> Understand and apply basic concepts of probability: understand and use appropriate terminology to describe complementary and mutually exclusive events;
Students will discover and use the complement principle: $P(A^c) = 1 - P(A)$	<b>S-CP.1.</b> Describe events as subsets of a sample space, using characteristics of the outcomes or as unions, intersections, or complements of other events.	<b>Grades 6-8</b> Understand and apply basic concepts of probability: understand and use appropriate terminology to describe complementary and mutually exclusive events;

### Prerequisites

Students will need to be familiar with finding the probability of a single event, A:

$P(A) = \frac{\text{Number of Outcomes for Event A}}{\text{Number of Total Possible Outcomes}}$ . Students will be familiar with the vocabulary associated with probability in the table below and have experience determining sample spaces and subsets. Students will be familiar with Venn diagrams.

#### Needed Vocabulary

Subset	A set that is part of a larger set.
Theoretical Probability	The number of ways an event can occur divided by the total number of possible outcomes.
Experimental Probability	Is based on repeated trials of an experiment or $(\# \text{ of Successes}) / (\text{Total } \# \text{ of Trials})$ .

## **Time Required**

This lesson will take place in two 50-minute lessons.

## **Materials and Preparation Required**

- Student Worksheet
- TI – Nspire Calculator or Core Math Tools
- Instructions for TI-Nspire Calculator Die Simulator or Core Math Tools (see Appendix)
- Two standard numbered cubes for each pair of students

## **Rationale for Lesson design:**

Students find it difficult to find the probability of compound events (Carpenter 1981). In a study of compound events where students summed up pairs of numbers, Fischbein et al. (1991) found that students do not possess any natural intuition when it comes to finding probabilities of compound events. Students may possess a tendency to globally evaluate the size of the sample space and its structure, and they do recognize the relationship between probability and the size of the corresponding sample space. From this they can develop a natural, intuitive tendency to evaluate the probability based on the sample space. Thus, evaluating the probability of compound events requires an understanding of the sample space (a pre-requisite that is revisited in this activity).

Shaughnessy (1977) found that conventional lecture may not be the best way to overcome student misconceptions of probability (such as the availability and representativeness heuristics presented by Kahneman & Tversky (1974)). Instead, he showed that student misconceptions could be addressed through hands-on experiments and activities in which students discovered probability concepts for themselves. He also showed that misconceptions can be discovered by having students compare conjectures to results obtained via experiments (1981). Furthermore, Shaughnessy and Zawojewski (1999) found that students need opportunities such as that presented in this lesson plan to work through substantive probability problems.

Finally, Beitzel, et al. (2011) found that students using only concrete representations such as Venn Diagrams to solve probability problems involving non-mutually exclusive events were outperformed by students instructed in using only the equation for the addition rule. Students that had to construct the Venn diagram to solve probability problems reported a higher cognitive demand than those using just the abstract representation of the formula. Thus, it is critical that students develop an understanding of the addition rules in order to be able to use them without constructing a Venn diagram.

The main author was unable to find any research on what students understand about the complement rule for probability nor how to teach this concept. As a result, we followed the idea of exploring Venn diagrams and discovering the Complement Principle by looking at subspaces.

# Odd or Even? The Addition and Complement Principles of Probability Teacher's Lesson Plan

## Describe the Context and Formulate a Question

Two people are playing a game of “Odd or Even” with two standard number cubes. (Each student should have the **Student Handout** in the Appendix). A trial occurs when two cubes are rolled once and points are assigned. The game consists of many trials, the number of which are decided upon by Player A and B before the first trial. Consider the following rules.

### **Odd or Even Rules:**

Roll two standard number cubes.

If the sum is 6 or an odd number, then Player A scores a point.

If the sum is 6 or an even number, then Player B scores a point.

Repeat the above steps an agreed number of times.

Whomever has the most points at the end of the game wins.

Ask students: “Suppose you play a game consisting of many trials. Predict which partner is likely to score more points in “Odd or Even”. Justify your reasoning.

Students will begin by making a prediction for the game “Odd or Even”. As students often possess misconceptions about probability, elicit responses to question 1 on the Student Handout to determine whether misconceptions about sample size exist, as developing intuition about events relative to sample size is often a misconception (Fischbein 1991). A common misconception is that a student is either Player A or Player B. Be sure to address this with students by emphasizing that we are analyzing the game, not playing it as the two players.

## Collect Data

Next, show students two six-sided number cubes and ask how they might determine who would win in a game of 1000 trials without actually playing 1000 times. To help address students’ misapplied confidence in small numbers, we will first look at samples of size 30 (Hope 1983). Before having students play the game (neither partner is Player A nor B), ask them what assumptions we are making about the game. Students should determine that we are assuming the number cubes are fair and that each side has the same probability of landing face up (we have a uniform probability distribution of the sides of the cube).

After students complete a game of 30 trials, bring the class together and ask what they found out. Begin to discuss whether 30 trials is enough. Did everyone get the same probabilities? Which ones are more accurate? The instructor may wish to pool the class data and ask whether students feel more confident about those results. Be sure to talk about underlying assumptions such as whether everyone tossed the cubes the same way. With the pooled data, students may still believe that we have enough trials to be fairly certain. How certain do we want to be? What if we wanted to predict the winner in a game consisting of 1000 trails? How could we do that without

actually playing 1000 times? Lead students to discover that it would be great if we could automate the process and use technology to perform many trials.

Question 4. Students will next simulate 1000 trials of the game “Odd or Even” using the TI-Nspire calculator or Math Core Tools (see Appendix A for directions on how to use both tools). After the simulation, have students compare their results with their original prediction (question 1) and their sample of size 30 (question 2). Give students the opportunity to change their original conjecture with an explanation. The process of moving from 30 trials to a class pooled data set to a simulation of 1000 trials, will help students combat the misconception of small numbers.

By the time students get to question 5 on the Student Handout, most will be skeptical that the game is fair. What does it mean to be fair? Which player would you rather be?

Before moving on, discuss/define the mathematical term “mutually exclusive” (two events are mutually exclusive if they have no outcomes in common). To help solidify the definition, ask students to describe some events in their everyday life that are mutually exclusive.

In the first part of the Explore section, have the students look at the three Venn diagrams. Before moving on, ask why two are labeled “not mutually exclusive” and one is labeled “mutually exclusive”. Then, discuss their similarities and differences.

## Analyze Data

The students will then compute the probabilities of two events that are mutually exclusive using simple probability and sample spaces. Next, students are asked to think about the relationships between  $P(A)$ ,  $P(B)$  and  $P(A \text{ or } B)$ . Students should discover that the sum of two probabilities is the third, and conclude that when  $A$  and  $B$  are mutually exclusive events,  $P(A \text{ or } B) = P(A) + P(B)$ .

Upon completing questions 7 – 11 on the Student Handout, make sure the students discuss their solutions first with their partner and then share out as a class. When sharing out as a class, define the addition rule of mutually exclusive events mathematically and write the definition in words the students understand. Sharing first with each other in small groups (in their native language, if necessary), is most important for the ELL students.

For Questions 12 – 17, students will find the probability of two events that are not mutually exclusive by using simple probability. They will then compare how they computed the probability of Player A winning to how they computed the probability of Player B winning. As part of this process, students will compare and contrast the results of mutually exclusive and not mutually exclusive events; they will conclude that if events are not mutually exclusive, then  $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \text{ and } B)$ . After completing question 12, have students share with their partner and then as a class. Upon completing questions 13 – 17, bring the class together to discuss what they discovered. When they share out as a class, define the addition rule for the probability of two non-mutually exclusive events mathematically and write it in words the students understand. Sharing first with each other in small groups (in their native language, if necessary), is most important for the ELL students.

Question 18 asks students to return to the original question – who would win in a game consisting of many trials? Bring students together as a class and have them compare the three probabilities that they have found: the one using dice and 30 trials of the game, the simulation of 1000 trials, and the theoretical probability. They should conclude that the simulation of 1000 trials of the game gave the most accurate experimental probability.

Once the addition rules are derived, the students will find the probability of the complement of Player A winning. A common misconception is that the complement of Player A winning must be player B winning. The way the game is designed, neither player may win: they could tie. They will then compute  $1 - P(A)$  and compare this result with the one found using simple probability. The students will then define the probability of the complement of an event A:  $P(A^c) = 1 - P(A)$ . Upon completing question 19 again have students discuss at their tables and share out as a class. Both ELL students and non- ELL students benefit by bouncing ideas off each other, and the discussion helps identify misconceptions. The instructor may choose to address misconceptions at the table-level or, if a misconception is prevalent throughout, address it as a class. Sometimes even if the misconception is not prevalent, it should be used for good classroom discussion to address non-examples.

## Interpret Results

Key points of the exploration that need to be discussed as a class are questions 11g, 17f, and 19g on the Student Handout. Following these key questions, the students will write in words and corresponding mathematical notation the two versions of the addition rule for probability and define the complement in the boxed areas. For ELL students, have them write in their native language or words that make sense to them along with the mathematical notation. It is vital that they understand the definitions for the Addition Principle for mutually exclusive and non-mutually exclusive events and the Complement Principle.

Question 11g. If two events, M and N, are mutually exclusive, then  $P(M \text{ or } N) = P(M) + P(N)$ ;  $P(M \cup N) = P(M) + P(N)$ ; Since M and N have nothing in common, or do not intersect, the events are mutually exclusive so you add the two probabilities together. “OR” is used to show that a sum may be in either A or B or in both, and is defined as the union that is notated by “ $\cup$ ” symbol.

Question 17f. If two events, H and I, are non-mutually exclusive, then  $P(H \text{ or } I) = P(H) + P(I) - P(H \text{ and } I)$ ;  $P(H \cup I) = P(H) + P(I) - P(H \cap I)$ ; Since H and I overlap they are non-mutually exclusive, and therefore we add the probabilities and subtract the probability of what they have in common (or the overlapping part) because it is included twice. “AND” is used to show the intersection of the two sets and notated by “ $\cap$ ” symbol.

Question 19g. If Q and R are complementary events, then  $P(R) = P(Q^c) = 1 - P(Q)$ ; Find the probability of the complement of Q, which is the probability of not Q and is written as  $P(Q^c)$ . Since 1 is the total probability of an event you can subtract a part,  $P(Q)$ , to obtain the remaining probability,  $P(Q^c)$ .

The final two questions ask students to reflect on the definitions (mutually exclusive, complementary) in relation to the Player A and Player B winning. They are not mutually

exclusive and they are not complementary. Students should be given time to think about these answers before developing their own game. A common misconception will be that either Player A or Player B wins, however, because they both score a point when the sum is 6, the game could end in a tie if an even number of rounds is played. In this case, neither player has more points than the other and neither wins.

### Suggested Assessment

1. A single letter is chosen at random from the word **MATHEMATICALLY**. Let A = “the letter E” and B = “vowel”. Are events A and B mutually exclusive? Explain, why or why not.
2. A standard number cube is rolled. Let event E be “an even number is rolled” and event O = “an odd number is rolled”. Are events A and B complementary? Why or Why not?
3. If the probability of Jack winning the Dinosaur Explosion game is  $\frac{1}{3}$  and the probability of Leo winning is  $\frac{1}{5}$ , what is the probability that either Jack or Leo will win the game? Support your answer with work.
4. Solve using two different methods. Make sure you show your work and explain the differences between your methods.  
The integers 1, 2, 3, ..., 20 are written on slips of paper and placed in a bowl and thoroughly mixed. A slip is drawn from the bowl at random. What is the probability that the number on the slip is either prime or divisible by 3?  
Method 1:  
  
Method 2:  
  
Explanation of the differences of method 1 and method 2:
5. Suppose F is the event that you draw a four and C is the event that you draw a club. Suppose that,  $P(F) = \frac{4}{9}$ ,  $P(C) = \frac{4}{9}$ , and  $P(F \cap C) = \frac{1}{3}$ . Find probability that you draw a four or a club.
6. On April 15,1912, the Titanic struck an iceberg and rapidly sank with only 710 of her 2,204 passengers and crew surviving. Data on survival of passengers (<http://www.encyclopedia-titanica.org/titanic-statistics.html>) are summarized in the table below.

	Survived	Did not survive	Total
<b>1st Class Passengers</b>	201	123	324
<b>2nd Class Passengers</b>	118	166	284

<b>3rd Class Passengers</b>	181	528	709
<b>Total Passengers</b>	500	817	1317

- What is the probability that someone on the Titanic was a 2<sup>nd</sup> or a 3<sup>rd</sup> class passenger?
- What is the probability that someone on the Titanic was a 2<sup>nd</sup> class passenger or survived?

### Assessment Answers:

- No, A and B are not mutually exclusive because E is a vowel and therefore A and B intersect.
- Yes, E and O are complementary because their union is the whole sample space and the events are disjoint.  $P(S) = 1$ , and  $1 = P(E) + P(O)$

$$3. \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

- Method 1: Counting principle or simple probability

List all the numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20

Count the numbers that are either prime or divisible by 3 without duplication  $14/20 = 7/10$

Method 2: Addition Rule for non-mutually exclusive events.

$P(\text{Prime or Divisible by 3})$

$$P(\text{Prime}) + P(\text{Divisible by 3}) - P(\text{Prime} \cap \text{Divisible by 3}) = 9/20 + 6/20 - 1/20 = 14/20 = 7/10$$

Method 3: Complements

You count the evens and take out the number 2 because it is a prime and then take out the numbers that are divisible by 3. That leaves you with probability of 6/20. Therefore

$$1 - 6/20 = 14/20 = 7/10$$

Possible explanation of the differences of methods:

The first method used the counting principle and the second method used the addition principle. With the counting principle, one counts the numbers in the set and with the addition principle, one finds the probability of a prime, the probability of a number divisible by three and the probability that a number is divisible by three and a prime. The addition rule is used to calculate the probability.

$$5. \frac{4}{9} + \frac{4}{9} - \frac{1}{3} = \frac{5}{9}$$

$$6. a. P(2^{\text{nd}} \text{ class or } 3^{\text{rd}} \text{ class}) = P(2^{\text{nd}} \text{ class}) + P(3^{\text{rd}} \text{ class}) = \frac{284}{1317} + \frac{709}{1317} = \frac{993}{1317} = \frac{331}{439}$$

$$b. P(2^{\text{nd}} \text{ class or Survived}) = P(2^{\text{nd}} \text{ class}) + P(\text{Survived}) - P(2^{\text{nd}} \text{ class and survived}) = \frac{284}{1317} + \frac{500}{1317} - \frac{166}{1317} = \frac{618}{1317} = \frac{206}{439}$$



## Possible Differentiation

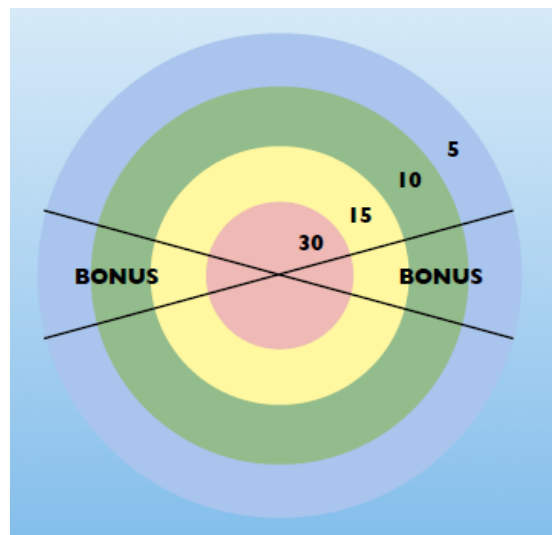
To extend this activity, consider looking at probabilities involving variables and geometric probabilities as in the following questions.

### Extension

- Suppose H is the event that you buy a two-story house and T is the event tan houses.
  - Suppose  $P(H) = \frac{2}{3}$ ,  $P(T) = \frac{3}{5}$ , and  $P(H \cup T) = \frac{13}{15}$ . Find the probability that you buy a tan two-story house.
  - Suppose  $P(H) = \frac{1}{2}$ ,  $P(H \cap T) = \frac{1}{4}$ , and  $P(H \cup T) = \frac{2}{3}$ . Find the probability that the house is tan.
  - Suppose  $P(H)$ ,  $P(T)$ , and  $P(H \cap T)$  can be represented by the following expressions respectively  $\frac{2}{x}$ ,  $\frac{3}{2x}$ , and  $\frac{1}{x}$ . Find the probability that you buy a house that is two-story or tan.
- What is the probability of scoring exactly 30 points with one dart thrown? The bonus region triples your score and each ring is 4 in. thick. The central angle for the bonus sector is  $30^\circ$ .

### Extension Solutions

- $P(H \cap T) = \frac{2}{5}$
  - $P(T) = \frac{5}{12}$
  - $P(H \cup T) = \frac{5}{2x}$
- Let A = "Area of 30 without Bonus"  
 Area of 30 without Bonus =  $\frac{300}{360} (4^2 \pi) = \frac{40}{3} \pi$   
 $P(A) = \frac{\frac{40}{3} \pi}{256 \pi} = \frac{5}{96} = .05208$   
  
 Let B = "Area of 10 with Bonus"  
 Area of 10 with Bonus =  $\frac{60}{360} (12^2 \pi - 8^2 \pi) = \frac{8}{3} \pi$   
 $P(B) = \frac{\frac{8}{3} \pi}{256 \pi} = \frac{1}{96} = .0104$   
  
 $P(A \text{ or } B) = P(A) + P(B) = \frac{6}{96} \text{ or } .0625$



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## Further Reading About the Topic

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# Odd or Even? The Addition and Complement Principles of Probability

## Student Handouts

### Launch:

Work with a partner to analyze the game “Odd or Even”. A trial occurs when two cubes are rolled once and points are assigned. The game consists of many trials, the number of which are decided upon by Player A and B before the first trial.

### **Odd or Even Rules:**

Roll two standard number cubes.

If the sum is 6 or an odd number, then Player A scores a point.

If the sum is 6 or an even number, then Player B scores a point.

Repeat the above steps an agreed number of times.

Whomever has the most points at the end of the game wins.

1. Suppose you play a game with a certain number of trials. Predict which partner is likely to score more points in “Odd or Even”. Justify your reasoning.
2. Next, ask your teacher for two standard numbered cubes. Then, play the game with 30 trials, keeping track of scores by tallying points:

Player A’s points:

Probability that Player A wins:

Player B’s points:

Probability that Player B wins:

3. Looking at your results, which player was more likely to win?

How do your experimental results compare with your prediction in question 1? Are your results surprising? In what way(s)?

4. Using the technology specified by your teacher, simulate tossing a pair of numbered cubes 1000 times. How many points did each player win?

Player A’s points:

Probability that Player A wins:

Player B’s points:

Probability that Player B wins:

How do your results using technology to simulate playing the game compare with the prediction you made in question 1 and your experimental result in question 3?

- Do you think the game is fair? Why or why not?
- We want to be sure of our answer to the previous question. What ideas do you have to find the exact/theoretical probability of each player winning?

**Explore:**

**On Your Own...**

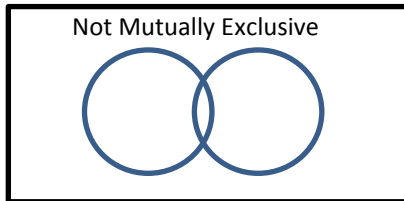


Diagram X

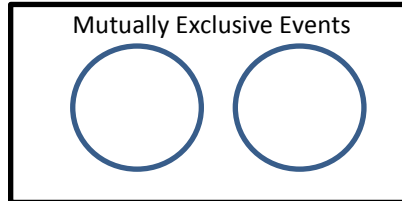


Diagram Y

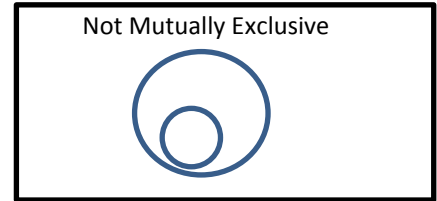


Diagram Z

- Consider the Venn Diagrams above where one circle in each diagram represents  $S =$  “the sum is 6” and the other represents  $O =$  “the sum is odd”.
  - The Venn diagram that represents Player A’s events is diagram \_\_\_\_\_. Explain why.
  - How is the probability of Player A winning related to the diagram you chose in the previous part?

**With your Partner...**

- The table to the right represents the sample space for the sum of the values when two cubes are tossed. Complete the table.
- Using the table to the right, shade the cells that represent Player A winning the game.

+	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

- What is the probability that Player A wins the game? That is, what is the probability of rolling a sum of 6 or a sum that is odd?

$$P(\text{Player A wins}) = P(S \text{ or } O) =$$

11. For each part, shade the corresponding event and use it to calculate the probability.
- a. Shade event “S”, and find  $P(S)$ .                      b. Shade event “O”, and find  $P(O)$ .

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

c. Compare the diagrams for “S” and “O” in the previous question with the diagram for “S or O” in question 9. What, if anything do you notice?

d. How do you think  $P(S)$ ,  $P(O)$ , and  $P(S \text{ or } O)$  are related? Explain your answer.

e. Test your conjecture to part d. by computing probabilities. Were you correct or do you want to change your conjecture?

f. Why do you think the relationship you found in part e. is true? Think about the event and sample spaces and the Venn diagram.

**g. Summarize**

If two events,  $M$  and  $N$ , are \_\_\_\_\_,  
 then  $P(M \text{ or } N) = P(M \cup N) =$  \_\_\_\_\_.

If two events,  $M$  and  $N$ , are \_\_\_\_\_, then  
 $P(M \text{ or } N) =$



**On Your Own...**

12. Now suppose we re-label the circles in the Venn Diagrams above so that one circle in each diagram represents  $S = \text{“the sum is 6”}$  and the other represents  $E = \text{“the sum is even”}$ .

a. The Venn diagram before question 7 that represents Player B’s events is diagram \_\_\_\_\_. Explain why.

b. Redraw the Venn diagram from the previous part. Shade and describe in your own words the region where  $S$  and  $E$  overlap.



c. How is the probability of Player B winning related to the diagram you chose in the previous part?

**With your Partner...**

13. Using the table to the right, shade the cells that represent Player B winning the game.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

14. What is the probability that Player B wins? That is, calculate the probability of rolling a sum of 6 or a sum that is even.

$P(S \text{ or } E) =$

15. Create two sample spaces and represent event “ $S$ ” and event “ $E$ ”.

16. What are  $P(S)$  and  $P(E)$ ?

$P(S) =$

$P(E) =$

17. Compare the diagrams of your sample spaces for the events “S” and “E” with that of “S or E”.

a. What do you notice?

b. Explain your discovery using the Venn diagram from question 12.

c. How do you think  $P(S)$ ,  $P(E)$ , and  $P(S \text{ or } E)$  are related? Explain your answer.

d. How is your answer in part c related to your answer from question 11d?

e. What is the probability that Player B wins?

$$P(\text{Player B wins}) = P(S \text{ or } E) =$$

f. **Summarize**

**If two events, H and I, are \_\_\_\_\_, \_\_\_\_\_, \_\_\_\_\_,**  
**then  $P(H \text{ or } I) = P(H \cup I) =$  \_\_\_\_\_.**

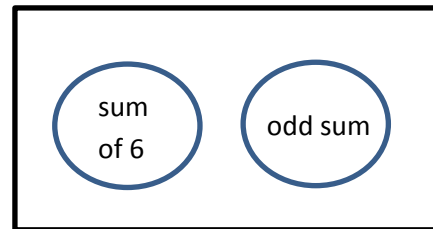
If two events, H and I, are \_\_\_\_\_  
then  $P(H \text{ or } I) =$  \_\_\_\_\_

Memorize  
Me!

18. We found the theoretical probability that Player A wins one point and the probability that Player B wins one point. Return to the original question: if you play a game with many trials, who is likely to win? Explain your answer.

**On Your Own...**

19. a. Shade the area of the Venn diagram to the right that represents the probability of not getting either a sum of 6 or an odd sum.



- b. Using the sample space table to the right, calculate the probability of getting a sum that is neither 6 nor odd using simple probability.

Mutually Exclusive

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P([S \cup O]^c) =$

**With your Partner...**

- c. Compare your sample space diagram for  $S \cup O$  (question 9) with the sample space diagram for  $[S \cup O]^c$  (question 19a). Compare also the Venn diagram for  $S \cup O$  (question 7) with the Venn diagram for  $[S \cup O]^c$  (question 19b). What do you notice?

- d. How do you think the probabilities  $P(S \cup O)$  and  $P([S \cup O]^c)$  are related? Explain your answer.

- f. Use  $P(S \cup O)$  to compute  $P([S \cup O]^c)$ .

**g. Summarize**

Two events A and B are called *complementary* if A and B are mutually exclusive events whose union is the universal space. Symbolically, we write  $A = B^c$  and  $B = A^c$ .

**If S and T are complementary events, then  $P(T) = P(S^c) =$  \_\_\_\_\_.**

If Q and R are \_\_\_\_\_,  
then  $P(R) = P(Q^c) =$  \_\_\_\_\_



20. Are the events of Player A winning a point and Player B winning a point mutually exclusive? Complementary? Support your answers using theoretical probabilities computed in previous questions.

**Close:**

21. Now it's your turn. Design a game using two six-sided number cubes so that each of Player A and Player B can win in one of two ways, but the two players do not have the same probability of winning. Calculate the probability that each player wins.



## Student Handout Solutions for Teacher

1. Partner A because the two events have nothing in common.
2. Answers will vary.
3. Answers will vary, but it is likely that most pairs of students determined that Player A won.
4. Answers will vary, but it is likely that with 1000 trials most pairs of students determine that Player A won.
5. Not fair because the outcomes will not be the same for partners A and B. A will have more because 6 is never an odd number and therefore the two conditions never intersect. Player B will have less because 6 is included as an even number.
6. Answers will vary.
7.
  - a. Diagram Y, because 6 will never be an odd number so the two events cannot overlap.
  - b. Because S and O do not intersect I can conclude that there should be more outcomes because of no overlap and that is why partner A should win. Partner A has a greater probability than Partner B.

8. Table shown to the right.

9. Table shown to the right.

10.  $\frac{23}{36}$

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

11.

a.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$P(S) = \frac{5}{36}$$

b.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$P(O) = \frac{18}{36}$$

- c. The diagrams for “S” and “O” combine to give the diagram for “S or O”
- d. It appears that  $P(S) + P(O) = P(S \text{ or } O)$  because the event spaces for S and O don’t have any overlap.
- e.  $P(S) + P(O) = \frac{5}{36} + \frac{18}{36} = \frac{23}{36}$  while  $P(S \text{ or } O) = \frac{23}{36}$ . They are equal, so our conjecture was correct.
- f. Because the event spaces do not intersect, their circles do not intersect in the Venn diagrams or are mutually exclusive.

- g. If two events, M and N, are mutually exclusive, then  $P(M \text{ or } N) = P(M \cup N) = \underline{P(M)} + \underline{P(N)}$ .

12.

- Diagram Z, because the number 6 is also an even number, therefore you know that it is contained in both sets S and E causing the overlapping of sets
- Since the sum of 6 is a sum that is an even number, then the event S is a subset of the event E, and therefore lies entirely within set E.
- Because of the intersection of S and E you cannot take the 6 into account twice. Therefore B's probability of winning should be smaller.

13.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

14.  $P(S \text{ or } E) = \frac{18}{36}$

15.

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Event E,

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Event S

16.  $P(E) = \frac{18}{36}$  and  $P(S) = \frac{5}{36}$

17.

- The two events overlap, in fact, event S is a subset of event E.
- We chose Diagram Z because S is a subset of E.
- We can't simply add the probabilities because we would be adding P(S) twice, once when we add P(E) and once when we add P(S).
- We need to subtract the intersection when events are not mutually exclusive:  $P(E) + P(S) - P(S \text{ and } E) = P(E)$  since  $P(S \text{ and } E) = P(S)$

e.  $P(\text{Player B wins}) = P(S \text{ or } E) = \frac{18}{36}$

f. If two events, H and I, are not mutually exclusive, then  $P(H \text{ or } I) = P(H \cup I) = \underline{P(H) + P(I) - P(H \cap I)}$ .

18. In 100 games, Player A is expected to win  $100 * \frac{23}{36} = 63.9$  points while Player B is expected to win  $100 * \frac{18}{36} = 50$  points. Thus, Player A is more likely to win.

19.

a. The region outside both circles is shaded.

b.  $P([S \cup O]^c) = \frac{13}{36}$

c. In both the sample space diagram and the Venn diagram, the events have no overlap and they make up the whole space.

d. The probabilities add to one.

e.  $\frac{23}{36}$

f.  $1 - \frac{23}{36} = \frac{36}{36} - \frac{23}{36} = \frac{13}{36}$

g. If Q and R are complementary events, then  $P(R) = 1 - P(Q)$ .

20. The events that Player A wins and that Player B wins are not mutually exclusive (they overlap because both can win a point if the sum is 6) and they are not complementary because they are not mutually exclusive (though their union is the sample space).

21. Answers will vary.

+	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
<b>1</b>	2	3	4	5	6	7
<b>2</b>	3	4	5	6	7	8
<b>3</b>	4	5	6	7	8	9
<b>4</b>	5	6	7	8	9	10
<b>5</b>	6	7	8	9	10	11
<b>6</b>	7	8	9	10	11	12

## Appendix

### TI-Nspire and TI-Nspire CAS Simulation Directions

Directions can also be found at

[http://math.kendallhunt.com/documents/daa2/cntns/daa2cntns\\_014\\_10.pdf](http://math.kendallhunt.com/documents/daa2/cntns/daa2cntns_014_10.pdf)

To simulate tossing two standard numbered cubes, we need to assume that each number has an equally likely chance of occurring. In other words, we need a uniform distribution of the numbers 1 through 6 so that we can use the random number generator in the List & Spreadsheet application.

1. Open the List & Spreadsheet app.
2. We have two numbered cubes and want to find their sum, so label columns A, B, and C “cube1”, “cube2”, “sumcube”, respectively.
3. To simulate rolling a six-sided cube 1000 times, type “randint(1, 6, 1000)” into the formula cell of columns “cube1” and “cube2”.
4. To compute the sum of the two cubes, in the formula cell for “sumcube” type “cube1 + cube2”. Then, press “Enter” to populate the column.
5. To see the distribution of the sums, you can create a histogram: from the Home screen, go to **Data & Statistics**.
6. Add “sumcube” to the x-axis. Select “Menu” and “PlotType | Histogram”
7. From the Histogram you can tally the number of outcomes that make up Player A’s and B’s outcome.

#### Core Math Tools Simulation Instructions:

Another way of making a simulation if you didn’t have access to the TI-Nspire calculator is using the program Core Math Tools. You can download it from the website <http://www.nctm.org/coremathtools/> by clicking on “downloadable suite”.

Once Core Math Tools is downloaded, open it and start a Simulation under Statistics & Probability.

1. From the column on the left, click the six-sided dice twice so we have two dice to work with.
2. Next highlight both die (on a PC this is done holding the “Control” key, while on a Mac you hold the “Command” key.)
3. Then, from the Build tab, click “Add” to simulate rolling two six-sided dice and adding the two together to show the sum.
4. We then want to see the number of times we roll the sum of an even number or a sum of 6. To simulate this, highlight the simulation, and click “Count # of” from the “Build” tab. A box will pop up with numbers 2-12. To count the number of even sums, you need to click on all the even numbers. First double click on the 2. Then, if you have a Mac, use “Command” while choosing “4”. Continue doing that until you have highlighted all of the even numbers. On a PC, do the same thing but use “Control” instead of “Command”.

- Then, to find the number of times we roll the sum of an odd number or a sum of 6, we repeat the process above (begin with step #1) selecting 3, 5, 6, 7, 9, 11.
- Finally, conduct 1000 Runs via the menu at the top. The simulation will show a “1” if the sum was even or 6 (odd or 6) and a “0” if not.
- To find the total count of sums for each simulation, select “Graph” and “Labels Bars” in the “View” drop-down menu.

