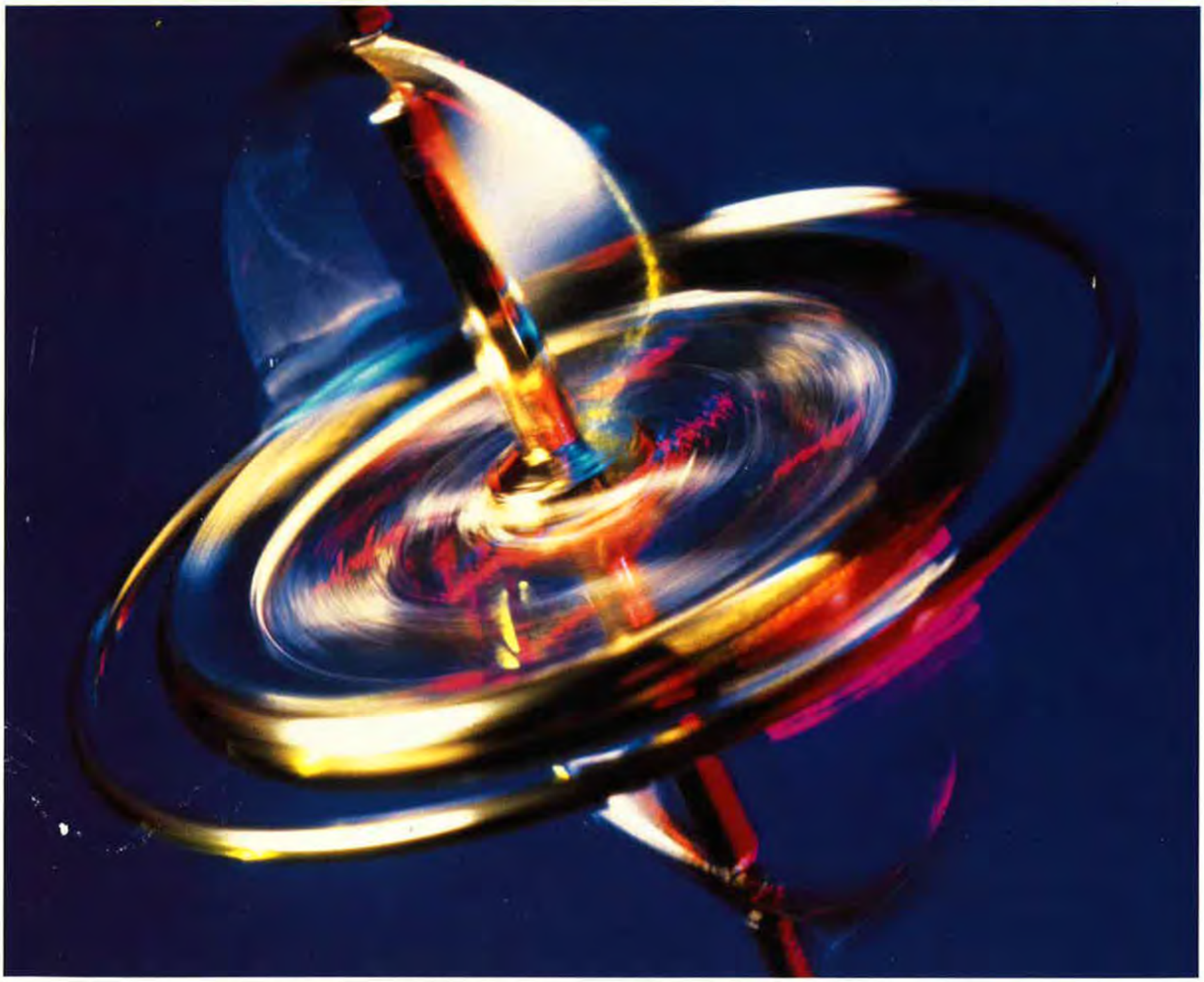
GEOMETRY

TEACHER'S EDITION

Exploring Centers

HENRY KRANENDONK AND JEFFREY WITMER

DATA-DRIVEN MATHEMATICS



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DATA-DRIVEN MATHEMATICS

Henry Kranendonk and Jeffrey Witmer

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About *Data-Driven Mathematics*

Historically, the purposes of secondary-school mathematics have been to provide students with opportunities to acquire the mathematical knowledge needed for daily life and effective citizenship, to prepare students for the workforce, and to prepare students for postsecondary education. In order to accomplish these purposes today, students must be able to analyze, interpret, and communicate information from data.

Data-Driven Mathematics is a series of modules meant to complement a mathematics curriculum in the process of reform. The modules offer materials that integrate data analysis with high-school mathematics courses. Using these materials will help teachers motivate, develop, and reinforce concepts taught in current texts. The materials incorporate the major concepts from data analysis to provide realistic situations for the development of mathematical knowledge and realistic opportunities for practice. The extensive use of real data provides opportunities for students to engage in meaningful mathematics. The use of real-world examples increases student motivation and provides opportunities to apply the mathematics taught in secondary school.

The project, funded by the National Science Foundation, included writing and field-testing the modules, and holding conferences for teachers to introduce them to the materials and to seek their input on the form and direction of the modules. The modules are the result of a collaboration between statisticians and teachers who have agreed on the statistical concepts most important for students to know and the relationship of these concepts to the secondary mathematics curriculum.

A diagram of the modules and possible relationships to the curriculum is on the back cover of each Teacher's Edition.

Using This Module

Exploring Centers is designed to integrate mathematical and statistical topics within a geometry class for high school students. The primary goal is to use measurements, shapes, distances, and principles of balance as a way to explore problems working with centers. Centers will initially strike students as merely a study of circles since their primary connection with this word is within that context. This module is designed to expand students' perspective of centers and to demonstrate that centers is a concept with a variety of interpretations and applications. Although the relationship to a circle is presented in the latter sections of this module, centers are also developed as numerical summaries of data, locations to balance weights, points or locations to equalize directed distances, locations to combine the impact of distance and weight, locations to minimize distances, or locations or points to minimize extremes. A center represents an attempt to be fair and to equalize the parameters important in special problems. For these reasons, a *center* is not an easy term to define. The use of the word becomes related to the specific problems investigated by the student. This module attempts to investigate several problems that are interesting and significant because of the connections to a center.

Why the Content Is Important

The mathematics incorporated in this module primarily involves using appropriate methods for summarizing data for generalizations and decision making. Several of the lessons require students to collect data; other lessons present data sets used by students to complete the problems. In all cases, the lessons guide students into organizing data, developing summaries (for example, the mean), and interpreting the results as directed by the larger questions or investigations. This process encourages students to investigate new types of problems and questions.

The mathematics used in the lessons is especially important because it enables students to understand the questions presented. Although many of the questions do not directly ask a mathematical question, students need to apply various mathematical topics to develop their solutions. The topics outlined in the *Mathematical Content* and *Statistical Content* become for students the tools in making decisions and explaining their solutions.

Mathematical Content

- Signed number operations
- Representation and interpretation of numbers on a number line
- Coordinate geometry applications:
  - Plotting points on a coordinate grid
  - Interpreting points from a coordinate system
  - Constructing coordinate systems

- Mathematical formulas
- Calculation of distances on a number line or coordinate grid

Statistical Content

- Calculation and interpretation of summary statistics
- Symbolic expressions for statistical summaries
- Interpretation of means as related to weight and distance
- Interpretation of medians
- Relationship of summary statistics to the interpretation of an application

Teacher Resources

At the back of this Teacher's Edition are the following:

- Quizzes
- Solution Key for quizzes
- Activity Sheets

Use of Teacher Resources

These items are referenced in the Materials section at the beginning of each lesson.

LESSONS	RESOURCE MATERIALS
Lesson 1: Problem 15a	• *Activity Sheet 1:* Data Summary 1 (for scale drawing)
Lesson 2: Problem 16 Problems 19 and 24	• *Activity Sheet 2:* Data Summary 2 (for "walking" activity) • *Activity Sheet 3: xy*-coordinate Grid 1 Unit I Quiz
Lesson 4	• Unit II Quiz
Lesson 5	• *Activity Sheet 4:* Triangle Options • *Activity Sheet 7: xy*-coordinate Grid 2
Lesson 6	• *Activity Sheet 5:* Quadrilateral Options • *Activity Sheet 7: xy*-coordinate Grid 2
Lesson 7	• *Activity Sheet 6:* Pentagon Model • *Activity Sheet 7: xy*-coordinate Grid 2
Lesson 8	• *Activity Sheet 6:* Pentagon Model • *Activity Sheet 7: xy*-coordinate Grid 2 • Unit III Quiz
Lesson 10 "The Big Picture" Project— Options 2 and 3	• *Activity Sheet 8:* U.S. Map (without coordinates) • *Activity Sheet 9:* U.S. Map (with coordinates and state capitals) • *Activity Sheet 10:* Data for U.S. Center Project (A—without coordinates) • *Activity Sheet 11:* Data for U.S. Center Project (B—with coordinates) • *Activity Sheet 12:* Summary of U.S. Center of Population • Unit IV Quiz
Lesson 12	• *Activity Sheet 13:* 3 x 3 and 5 x 5 Grids for Stores
Lesson 13	• *Activity Sheet 13:* 3 x 3 and 5 x 5 Grids for Stores
Lesson 14	• *Activity Sheet 13:* 3 x 3 and 5 x 5 Grids for Stores • Unit V Quiz

Where to Use the Module in the Curriculum

This module is designed for use primarily within a geometry class or within a mathematics class incorporating the study of shapes and plane geometry topics. The lessons are developed to provide interest in statistical problems involving data sets collected and generated from geometric shapes and related topics in a geometric investigation.

This module includes five units. Each unit contains two to four lessons. The Pacing/Planning Guide indicates that several of the lessons ideally fit at the beginning of a typical geometry class, while other lessons should be used after preliminary work with geometric topics has been developed. This module can be used throughout an academic year in which larger investigations involving data and geometric inquiries would enhance the course.

Prerequisites

Students should have worked with signed number operations and general measurement problems before starting this module. Other sections require previous work with triangles, quadrilaterals, and other polygonal shapes. The detail of the prerequisite work with shapes is not extensive and should be covered in most 9th- or 10th-grade geometry classes. Some of the problems and investigations are quite involved and tedious. Group work is especially effective for those problems. As expected, however, completion of the tasks for these problems is dependent on students' willingness to work together in small groups.

Pacing/Planning Guide

The 14 lessons included in this module can be completed at various times in a geometry course or in a similar mathematics course. Most lessons are designed to be completed in 2 or 3 class sessions. The prerequisite skills described in the following table are general descriptions of the most important skills expected of students at the beginning of the unit. As indicated in the table, some of the lessons assume that identification and classification of geometric shapes have been previously learned by students. Also, some lessons assume familiarity with important characteristics of common shapes so that investigations and problems are more manageable. (For example, familiarity with the medians of a triangle or the diagonals of a parallelogram are important in Unit III. This is included in the Unit III summary.) Several skills not listed in this table are necessary in the lessons; however, they are developed as part of the objectives of the lessons and are not considered prerequisites.

Pacing/Planning Guide

The table below provides a possible sequence and pacing of the units.

UNIT	PREREQUISITE SKILLS	PACING (number of class sessions)	TIMETABLE
Unit I: Estimating Centers of Measurements	• Measuring distances with a ruler • Operational skills with signed numbers	(2 lessons) a total of 3 to 4 sessions	Beginning of a geometry class
Unit II: Centers of Balance	• Plotting and interpreting points on a number line • Operational skills with signed numbers	(2 lessons) a total of 4 sessions	Beginning of a geometry class
Unit III: "Raisin" Country	• Identification of angle and side descriptions of triangles • Identification of the definition of medians of a triangle • Identification of polygons by the number of sides and descriptions involving parallel lines, supplementary angles, diagonals, and corresponding angles • Plotting and interpreting points in a coordinate system	(4 lessons) a total of 6 to 8 sessions	Middle of a geometry class after prerequisite skills have been developed through work with triangles, quadrilaterals, and polygons
Unit IV: Population Centers	• Prerequisites similar to those of Unit III	(2 lessons) a total of 4 to 6 sessions (Lesson 10 involves a project.)	Middle of a geometry class Use right after Unit III since several references to examples and illustrations in Lessons 5–8 are incorporated.
Unit V: Minimizing Distances by a Center	• Identification of terms related to circles • Plotting and interpreting points in a coordinate system	(4 lessons) a total of 6 to 8 sessions	Middle to end of a geometry class; recommended after general work with circles has been developed

Technology

A graphing calculator similar to a TI-83 is needed for most of the lessons. Several of the lessons, particularly beginning with Unit III, would be well-supported with spreadsheet software. The calculator must have List capability. The linked cells of a spreadsheet offer a number of options in working with the data.

An overhead projector will be helpful. Overhead transparencies of particular data sets, graphs, or *Activity Sheets* can be useful during class discussion.

Grade Level/Course

This module is intended for a 9th- or 10th-grade geometry course or for a mathematics course involving investigations of geometric shapes. Although this module was designed to complement the geometry curriculum, it is also appropriate to use in a variety of courses designed to develop projects and work with data.

UNIT I

Estimating Centers of Measurements

LESSON 1

Centers of a Data Set

Materials: tape measures and/or meter sticks, *Activity Sheet 1*
Technology: graphing calculator
Pacing: 2 class sessions

Overview

Lesson 1 quickly introduces students to the data summaries of mean, median, and mode. This introduction assumes students are aware of these terms and might have previously worked with them. Mean, median, and mode are not introduced as centers but rather as possible data summaries that might qualify as a measure of center. This is an important point as the concept of a center is embedded in the nature of the particular problem.

This lesson develops an activity carried out by a geometry class at Rufus King High School and uses the data collected by the students. Each data set collected requires a summary in order to develop a scale drawing. The steps to implement this activity are outlined to help your students organize a similar investigation at your school site. This outline is provided in the student's lesson.

Teaching Notes

This lesson uses the familiar terms of *mean, median,* and *mode* to describe the collected data. Mean and median are the primary topics; however, a useful application of the mode is suggested. Emphasis of the mean and median is important since the primary applications of center in the lessons that follow are applications of these descriptions of centers.

This lesson does not conclude with a precise definition of center. This observation is intentional as the lesson is designed to encourage further investigations in other lessons and modules of this series. A need for creating a center is based on the importance of communicating a summary of a collected set of data. This first lesson bases the importance of a center by using measurements and the variance of estimates resulting from this process.

Technology

As indicated, a scientific or graphing calculator is important to complete the numerous and often tedious calculations of the group activities. The type of calculator is not critical in this lesson; however, the List options of a calculator similar to a TI-82 or TI-83 could be applied to several problems. The format for using the lists is suggested by the organization of the data sets in columns. This option, however, is not required and therefore is not highlighted in the lesson.

Follow-Up

A suggested follow-up for this lesson is outlined in the Further Practice section of this lesson. The field-test groups particularly enjoyed this extension and developed several small-group activities to highlight the local application of this lesson. This moved the lesson from the Rufus King High School model to the local school.

STUDENT PAGE 3

LESSON 1

Centers of a Data Set

What is meant by a "center"?

How is a "center" calculated and interpreted?

The number representing your grade point average is a center, as well as the average number of students in a class or the average number of students arriving late to school. Would you consider the location of your locker as "centrally" located?

How would you determine this central location provided that was an important concern?

OBJECTIVES

Calculate the mean, median, and mode of a data set.

Interpret a center as an estimation of a data set.

Visualize the accumulation of error resulting from estimates of measured data.

Although several examples of centers will be defined as these lessons are developed, examples based on the *best* estimate of a set of data will start the discussion of centers. The importance of this type of center is demonstrated using the backup times collected at a swim meet. Whenever possible, an electronic clock is used to record a swimmer's official time. If, however, this clock does not function correctly, which can easily happen due to equipment problems or a poor "touch" by the swimmer, three backup timers are used. Surprisingly, it is rare to have the same times reported by the three backup timers. If three backup times are used, how do you think the official time is determined for a swimmer?

INVESTIGATE

Swim Meets

At a regional swim meet, Kristin, Melissa, and Shauna were involved in the same heat of the 50-yard freestyle. The top swimmers from each heat moved to the next level of competition. Their official times were used to both qualify them for the

Solution Key

Discussion and Practice

1. Answers will vary. Unless a student knows how times from a swim meet (or similar athletic events) are used, an explanation could be made for any one of the swimmers to be declared the winner. Obviously, this is the point of the problem.
2. Similar explanation could be made about a swimmer losing the heat as winning. See above explanation for problem 1.
3. Melissa
4. Answers will vary, however, the fact that Melissa could be the winner based on one time (and ignoring the slower times that would indicate she lost the heat) suggests this may not be the best criteria for summarizing the set of times.
5. Answers will vary. The purpose of this problem is to have students speculate on what is the best method to summarize a set of numbers. The interesting feature evident in the above swim times is how various methods of summarizing numbers could change the outcome.
6. Yes. The discrepancy in the timers indicates an unusual feature. One timer might have started late giving her a better swim time, or two timers might have started early giving her a slower swim time, or some other time exists but was not obtained by this group of timers. Note the range of values in Melissa's set.

STUDENT PAGE 4

next competition and provide their placement in a heat of this event. As a result, the official time for each swimmer was very important! Although an electronic clock was used to record the official times at this particular meet, the following times were recorded by the backup timers:

Event: Girls' 50-yard Freestyle

Swim Club Record: 27.0 seconds

(All times recorded from the backup timers are in seconds.)

	Kristin	Melissa	Shauna
Timer 1	27.2	27.4	27.3
Timer 2	27.3	27.4	27.4
Timer 3	27.1	27.0	27.1

Discussion and Practice

1. Of the three swimmers, who probably won this swim heat? Why?
2. Of the three swimmers, who probably lost this heat? Why?
3. If the criterion for determining an official time was the best time recorded by the three timekeepers, who would benefit the most?
4. Do you think selecting the fastest time from the three times recorded for each swimmer is a fair method for determining an official time? Explain why or why not.
5. What method would you suggest for determining an official time if backup times were needed? Explain why you think your suggestion is fair.
6. Do you think the times reported for Melissa are unusual? Explain.

Mean, Median, and Mode

An official time for each swimmer could be determined in several ways. Three frequently used summaries for a data set are referred to as ***mean***, ***median***, and ***mode***. Each of these three summaries should be considered when selecting a center or description of a data set.

The ***mean*** of a data set is also called an ***arithmetic average***. It is calculated by finding the sum of the data and then dividing

STUDENT PAGE 5

7. **a.** The mean represents a value that is between the high and low values recorded.

b. Generally yes; it fits between the extremes.

c. Although the mean represents a good middle value, it is also time-consuming to determine under usual swim meet conditions. As a result of the problem of calculating and recording this value during a typical meet, this summary value is not used. When backup times are required, a quick and easy method is needed to report as the official time.

by the number of members in the set. It is the most common summary of a set of numbers.

7. The following process determines the mean of the three times reported for Melissa:

$$\frac{(27.4 + 27.4 + 27.0)}{3} = \frac{81.8}{3} = 27.2\overline{6}$$

a. Describe how a mean represents a "center" value of this data set.

b. Does the mean seem to be a fair value of Melissa's swim time?

c. Would you recommend the mean as the official time of a swimmer if the electronic time was not accurate? Explain.

The *median* is another summary of data. Generally it is described as the "middle value" of an ordered data set. This middle value is most easily determined if the number of values belonging to the data set is odd. In those cases, the median is an actual value of the data set. If, however, the data set has an even number of values, then the median is the mean, or arithmetic average, of the two data values "centered" around the middle of the set. Following is an example of finding the median for a data set of six values:

Data set: 34, 42, 16, 30, 40, 45
Ordered data set: 16, 30, 34, 40, 42, 45

The median ("middle" value) is the mean of the third and fourth values:

3rd value 4th value

16, 30, 34, 40, 42, 45

$$\frac{(34 + 40)}{2}$$

$$= \frac{74}{2} = 37$$

STUDENT PAGE 6

8. Similar steps are used to find the median of Kristin's three times. Arrange (27.2, 27.3, 27.1) in ascending order:

27.1 27.2 27.3

The middle or median is the second member of the ordered values:

27.1 27.2 27.3

↑

The median

a. Describe how a median represents a "center" value of this data set.

b. Does the median seem to be a fair representation of Kristin's time?

c. Examine Melissa's and Shauna's recorded times also. Would you recommend the median as the official time of a swimmer? Explain.

9. Calculate the mean of Kristin's times.

a. What is unusual about Kristin's data set when finding the mean and median?

b. Why does this happen in part a?

The *mode* is defined as the "most frequent data value" or values. A mode exists only if a specific data value in the data set occurs more than once. If several values reoccur in an expanded set of numbers, then the mode is the most frequent value. It is possible for a data set (with more than three values) to have more than one mode. When all data values occur only once in a data set, then the data set is considered to have no mode.

10. Of the three swimmers, who would be able to have her official times summarized by a mode?

11. The mode does not actually describe a "center" value for a data set of three swim times. Explain this statement by using the times for one of the swimmers.

12. If a mode exists for a data set of three values, it is similar to what other data summary, the median or mean? Demonstrate this by developing another set of three numbers.

13. Consider a set of four or more numbers in which at least one mode exists. Would the mode of this expanded set of numbers always have the same feature you summarized above? Explain your answer or demonstrate with a data set.

8. **a.** At least in this example, it is between the highest and lowest values recorded for Kristin.

b. Yes.

c. Shauna, like Kristin, has a good middle time with the criteria of the median. Melissa, however, does not have her times summarized by a "center" value using the median. This again highlights the usual feature of the collected times for Melissa.

9. **a.** Kristin's mean is 27.2 seconds, which is also the median.

b. This happens as 27.2 seconds is in the middle of the other two times. Also, the other values have the same number of seconds above and below this middle value. Essentially 27.2 is the average of the three times.

10. Melissa; Melissa is the only swimmer who had two of her recorded times the same.

11. If a mode exists for a data set of three values, it is either the largest value or the smallest value of the set. (It cannot represent a middle value.)

12. If a mode exists for a data set of three, it is the same as the median.

13. No. A data set of four or more will have a mode if two or more data values are equal. If two data values are equal out of four, then the mode could be the largest, the middle value, or the lowest value of the set. Although usual, the mode can also be described by more than one value. For example, the set {25, 25, 26, 26} is considered to have two modes, 25 and 26. This feature can be even more noticeable in data sets represented with more than 4 values. For this reason, the mode is not a very useful description of a data set.

STUDENT PAGE 7

14. a. Generally the median is a good middle value, although an exception was noted in Problem 12. The selection of the median is generally summarized for a set of three values as "throw out the high and the low"; this leaves a middle value for the official time. In a practical sense, the median is the easier summary to determine from a group of three timers. Frequently swim heats are run back to back, therefore, a quick and relatively accurate time is needed by the swim officials.

b. If there are three timers, then the median is very easy to determine (this again highlights the need for a quick and easy method to determine the swimmer's time). Expanding the number of timers will make this process more involved. If only two timers are used, then the official time is also the mean of the times recorded.

14. Generally swim meets use the median time as the official time of a swimmer when a backup time is needed.

a. Why do you think this value is used?

b. Officials attempt to obtain three backup timers. Why? How would the situation change with two timers, assuming they use the median? How would it change with four timers?

Centers of Measurements

Students in a geometry class at Rufus King High School were asked to develop a scale drawing of the first floor north entrance hallway. This drawing was to be used to determine measurements and calculations for purchasing floor tiles, bulletin boards, and so forth. Estimates of costs for each of these projects require an accurate drawing of this hallway. This project was developed by the class in the following way:

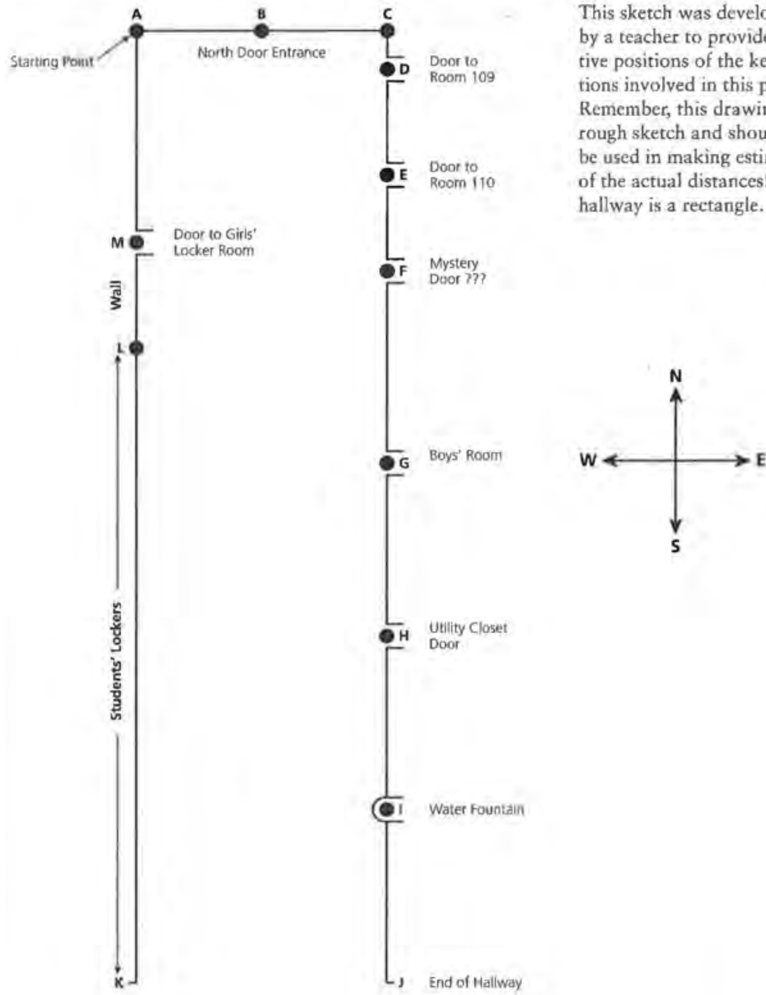
Step 1. Key locations along the hallway floor were marked and labeled with masking tape.

Step 2. A rough sketch of the hallway was developed to highlight the key locations.

Step 3. Seven groups of students were formed. Each group was responsible for recording the measurements obtained on the Data Summary Sheet.

Step 4. Using the data collected from the seven groups, each group was responsible for developing a scale drawing of this rectangular hallway.

STUDENT PAGE 8



This sketch was developed by a teacher to provide relative positions of the key locations involved in this project. Remember, this drawing is a rough sketch and should not be used in making estimates of the actual distances! The hallway is a rectangle.

15. a. Errors in measuring, differences in measuring tools, slight differences in starting and ending points of the distances measured, or other answers could be summarized by students. Students might be asked this question again after they attempt the Further Practice section as the actual process of measuring distances helps them understand the potential for errors.

b. Completing the values for this table is rather tedious. This problem takes quite a bit of time. As a result, this problem is ideal for group work.

STUDENT PAGE 9

Data Summary for the Measurement Experiment
Rufus King High School North Entrance Hallway (meters)

Group	AB	BC	CD	DE	EF	FG	GH
Group 1	1.78	1.77	0.65	8.34	1.87	5.02	2.10
Group 2	1.78	1.75	0.58	9.35	1.85	4.98	1.75
Group 3	1.77	1.75	0.59	9.20	1.88	4.95	2.25
Group 4	1.78	1.76	0.66	9.10	1.90	5.03	1.95
Group 5	1.75	1.80	0.67	9.05	1.89	4.95	2.23
Group 6	1.68	1.76	0.63	9.10	1.88	4.80	2.15
Group 7	1.78	1.76	0.66	8.91	1.87	5.10	2.17
Mean							
Median							
Mode							

Group	HI	IJ	JK	KL	LM	MA
Group 1	3.34	3.08	3.55	8.88	5.30	11.32
Group 2	3.96	3.50	3.65	9.52	5.00	10.91
Group 3	3.55	3.64	3.56	9.14	5.62	12.12
Group 4	3.12	2.68	3.72	8.78	5.45	11.52
Group 5	3.05	2.85	3.43	8.68	5.15	13.58
Group 6	3.22	2.98	3.31	8.94	5.23	9.12
Group 7	3.40	3.05	3.33	11.25	5.35	11.56
Mean						
Median						
Mode						

15. Using the class measurements, answer the following questions before developing a scale drawing.

a. Review the data sets. Each group was expected to measure the same distances. Why are the recorded measurements different?

b. Complete the data sheet by determining the means, medians, and modes for each of the distances labeled. Copy and complete this part of the data sheet.

c. You will be developing a scale drawing of this rectangular hallway. What do you anticipate to be the main problem in constructing an accurate sketch of this hallway?

16. Consider the following four criteria in selecting the "best" center of the measurements reported by the groups.

- the mean of the values for any of the specified distances
- the median of the values

Data Summary Sheet for the Measurement Experiment
Rufus King High School North Entrance Hallway (meters)

Group	AB	BC	CD	DE	EF	FG	GH
Mean	1.76	1.76	0.63	9.01	1.88	4.98	2.09
Median	1.78	1.76	0.65	9.10	1.88	4.98	2.15
Mode	1.78	1.76	0.66	9.10	1.87, 1.88	4.95	none

Group	HI	IJ	JK	KL	LM	MA
Mean	3.38	3.11	3.51	9.31	5.30	11.45
Median	3.34	3.05	3.55	8.94	5.30	11.52
Mode	none	none	none	none	none	none

c. The primary concern is to determine what measure or what summary of the measures should be used to develop the scale drawing.

STUDENT PAGE 10

16. Students should be encouraged to select their own values and justify their selection. The following values are used to develop a scale drawing of the hallway and provides an example of the process described in this lesson. The measures used for each segment will vary from student to student (or from group to group).

- the mode of the values (if one exists)
- an "average" (mean or median) of a subset of the values

(The last criterion allows some measurements to be thrown out as obvious errors. The measurements remaining would then be averaged or "centered.")

a. Determine the measurements you will use to develop the scaled sketch of the rectangular hallway. Explain what criterion you used to select this best estimate and why. Complete the following table which is also on *Activity Sheet 1*.

Measurement of segment to be used in your sketch	Criteria used for this measurement	Why?
AB =	______	______
BC =	______	______
CD =	______	______
DE =	______	______
EF =	______	______
FG =	______	______
GH =	______	______
HI =	______	______
IJ =	______	______
JK =	______	______
KL =	______	______
LM =	______	______
MA =	______	______

b. Develop a scale drawing of this hallway based on the measurements selected in your table. Begin developing this scale drawing by placing the starting point A in the upper-left corner of a blank sheet of paper. (You might consider developing this sketch on legal size paper as this hallway is quite long.) Measure and mark each of the labels provided in the teacher's sketch of this hallway. Use a scale of 1 cm = 1 meter or a comparable scale.

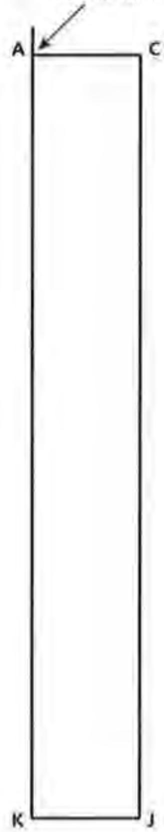
(16) a.

Measurement of Segment to Be Used in Your Sketch	Criteria Used for This Measurement	Why?
AB = 1.78	Mode or median	Although not a center value of the numbers, this value is a good representation of the set of numbers.
BC = 1.76	Mean, median, or mode	A good, solid center of the data set.
CD = 0.65	Median	A good center description of the data set.
DE = 9.10	Median or mode	The median centers the large differences in the data set.
EF = 1.88	Mean, median, or mode	A good, solid center of the data set.
FG = 4.98	Mean or median	Value is a good center of the data set.
GH = 2.15	Median	Recorded values represent quite a range—median represents a good balance.
HI = 3.38	Mean	A good center of the recorded data set.
IJ = 3.05	Median	This set has quite a range—this value centers the set of numbers.
JK = 3.55	Median	A good center of the varied values of this data set.
KL = 8.99	Throw out 11.25 m and calculate the mean of the remaining set of numbers.	11.25 m appears to be an error in measurement or recording, and it should not be considered in the final value of KL.
LM = 5.30	Mean or median	A good, solid center.
MA = 11.84	Throw out 9.12 m and calculate the mean of the remaining set.	9.12 m appears to be a mistake—center the remaining values.

b. The following drawing is a rough sketch of this hallway using a smaller scale than indicated to the students. Primarily observe the accumulated errors in measurements result in the final segment not forming a rectangle. The idea presented in this drawing is that the best estimates of the segments should produce a fairly good approximation of the rectangular layout of this hallway.

None of the scale drawings will form a perfect rectangle! (If you get one, the student did not follow the directions of this activity!) Each measurement is an estimate, however, some estimates are better in developing a fit of the segments to form a rectangle. The need for a relatively accurate drawing is highlighted by the problems related to estimating costs of installing tiles and an announcement board.

Note how segments do not meet to form the rectangle.



17. The individual segments will not fit together to form the rectangle. Either the final segment is under or over the value needed to make a fit of the segments as a rectangle.

18. Total measures of the widths and lengths should be equal. Comparing widths indicates AC = 3.54 m and JK = 3.55 m. Comparing lengths indicates CJ = 25.19 m and KA = 26.13 m. Differences are expected given the way the measurements were made.

19. The larger segments show more variation and are probably more subject to error. For example, segments MA, KL, and DE are more difficult to estimate given the variations noted in the measurements.

STUDENT PAGE 11

17. Your drawing represented the best measures of the distances given your criteria. As you compare your sketch to other members of the class, how does each sketch indicate the measures are not "perfect"?

18. How could the fact that the hallway is rectangular help in developing an accurate sketch?

19. Of the measurements recorded on the data sheet, what specific distances do you think are the most likely to be inaccurate. Explain why you think this.

20. A group of students studying the collected data commented that the Rufus King students obviously used meter sticks (rulers) instead of a measuring tape. Would you agree? Explain your answer.

SUMMARY

Collected data frequently needs to be summarized by a center. Mean, median, or mode are summaries of a data set representing its "center" or summary.

Practice and Applications

21. From your sketch, determine the following distances:

a. the distance (to the nearest meter) from the door of room 109 to the girls' locker room door.

b. the distance from the water fountain to point A.

c. the distance from the boys' room door to the girls' locker room door.

22. If floor tiling will be installed at a cost of \$16.75 per square meter, what is your estimate of the total cost of installing floor tiles for the entire hallway?

23. An announcement board for the athletic department will be installed from point A to a point 1 meter north of the girls' locker room door (or 1 meter north of point M). If your scale drawing is used to estimate the length of the board, what problem do you encounter with this last segment of the drawing?

24. If the announcement board discussed in problem 23 costs \$10.70 per linear meter, what is the projected total cost of this project? (The announcement board has a standard height, therefore, your estimates do not need to involve that dimension of the board.)

20. This conjecture is based on the more noticeable differences in the larger measurements, something more likely when placing meter sticks back to back. Students might be interested in testing out this idea. Mark a distance of at least 10 meters and have three or four groups measure the distance. Collect estimates from each group based on the use of a meter stick and a tape measure.

Practice and Applications

21. The distances provided by the students should be based on their scaled drawing. The following distances are approximations based on the measurements used in problem 15:

a. Students are measuring the segment DM from their scaled drawing. This distance is 10.4 cm, therefore, representing 10.4 m. To the nearest meter, this is 10 m.

(21) **b.** Students are measuring the segment IA from their map. This distance approximately represents 21.9 m, or to the nearest meter, 22 m.

c. Students are measuring GM. This distance is approximately 7 m.

22. The major step in this problem is to estimate the area of the hallway. In addition, an estimate of area is based on a student's estimate of the length and width of the hallway. Given the scale drawing used as an example of this process in this teacher's edition, a width measure of 3.55 m and a length measure of 25.7 m (the mean of the lengths CJ and KA) would given an area of

$(3.55 \text{ m})(25.7 \text{ m}) = 91.24 \text{ m}^2$.

Therefore, the total cost for tiling the hallway would be

$(91.24 \text{ m}^2)(\$16.75) = \$1528.27$.

23. MA is the segment that clearly does *not* meet or fit! (It is the segment that generally gives students a sense that "something is wrong"). It is the segment that indicates how estimated values are not totally accurate. If an estimated value for the board is discussed, students will probably use the summary value of MA from the chart, or MA – 1 m = 11.84 m – 1m = 10.84 m.

24. $(10.8 \text{ m})(\$10.70) = \$115.56$

Again, base this estimate using the measures recorded by the student.

STUDENT PAGE 12

The following suggested problems follow the process described in the investigations of this lesson by the Rufus King students.

25. As a class, identify a hallway or room in your school that could be used to develop a scale drawing.

a. Select key locations along the perimeter of this room or hallway. Either with masking tape or paper, identify the key locations.

b. Develop a sketch of the room or hallway using the key locations.

c. Design a data sheet to record the measurements indicated in the sketch.

26. Form several groups to determine measures of the distances. Each group should complete the following steps:

a. Using the same type of measuring tool (for example, meter stick or tape measure), measure and record the values designated in the class data sheet.

b. Estimate the distances designated in the sketch using some criterion of centering.

c. Design a scale drawing of the room or hallway.

27. Is the room or hallway selected for this class project a rectangle, square, or other shape?

How can the shape of a room help you determine the accuracy of the scale drawing?

28. How could each group test the accuracy of its scale drawing?

Further Practice

25.–27. Problems 25 to 28 follow the format of problems 1–24 of this lesson. The data should be collected by the students. The room or hallway identified for this practice should have key locations identified to encourage measuring and estimating skills.

28. Accuracy is difficult to determine in this type of activity (the point of the lesson!), however, the "fitting together" of the measured segments indicates a way to examine the accuracy of the various estimates.

LESSON 2

Descriptions Through Centers

Materials: tape measure, *Activity Sheets 2 and 3,* Unit I Quiz
Technology: graphing calculator
Pacing: 1 to 2 class sessions

Overview

Lesson 2 continues the description of center as a summary of data and develops additional activities to get students involved in collecting data. This lesson works, however, on another dimension—namely the possible connection of two data sets. This connection could be expanded through a study of linear relations and a best-fitting line. These extensions are given a more thorough treatment in other modules of this series. This lesson primarily asks students to summarize the relationship suggested by collected data. This may seem a general treatment of scatter plots for students who have previously worked with them. The precision some students might expect is not developed in this lesson because the primary goal is to emphasize the importance of data summaries (i.e., centers) and not linear relations.

Teaching Notes

This lesson extends the type of applications requiring a summary of data described as a center. This lesson complements the first lesson by highlighting problems involved in communicating the results of a collection of data. This lesson was particularly important in demonstrating how different centers could be used to summarize data and produce varying conclusions about the data.

Preparation is recommended in determining a location for students to collect the data for walking the 50-m distance. Also, a tape measure used for recording students' heights should be set up before the lesson is attempted.

Technology

Several questions require tedious calculations. As indicated in Lesson 1, this process requires the use of a scientific or graphing calculator. At this point, the problems do not require the special features of a graphing calculator. The lesson concludes with a look at a scatter plot of data. The lesson directs the students producing this scatter plot to use the grids provided by the activity sheet. However, implementing this scatter plot using a graphing calculator could be developed. The format of producing the graphs is a teacher decision. (The use of scatter plots for the investigation of linear relations is presented thoroughly in other modules of this series. This lesson reviews these topics but does not give a formal presentation of linear regression topics.)

Follow-Up

This lesson suggests several follow-up activities within the lesson. These activities can be modified or developed depending on the students abilities in completing the stated objectives. The quiz for Unit I is designed to review the collection activities developed in the lessons and the process of summarizing the data using a center. The assessment, or quiz, in this particular unit is designed to follow a format similar to the problems presented in the lessons.

STUDENT PAGE 13

Solution Key

Discussion and Practice

1. **a.** The coordination described in the problem maximizes the jumper's momentum and ability to put forces together for obtaining distance.

b. Jumper's height, physical development or conditioning, leg length, or similar answers of this type are appropriate.

LESSON 2

Descriptions Through Centers

Have you ever watched athletes competing in track and field events?

What kind of special preparation would be necessary for competing in an event such as the broad jump?

Track and field events require athletes to coordinate running and jumping skills. Athletes participating in the high or low hurdles spend considerable time counting the number of steps needed to reach a position to begin the jump over a hurdle. Incorrectly counting the number of steps can throw off the coordination and the resulting time for the athlete to complete the event. Similarly, the broad jump event in track and field also requires athletes to coordinate running and jumping.

OBJECTIVES

Calculate the mean, median, and mode of a data set.

Interpret a center as an estimation of a data set.

Interpret a center as a description of an individual's physical characteristics.

INVESTIGATE

Track and Field Events

Olympian Carl Lewis carefully prepared for several summer Olympics (and subsequent world records in the broad jump) by running a specific distance and counting his steps or strides before making the actual jump. How might Carl determine the number of steps to the jumping-off line?

Discussion and Practice

1. Consider the broad jump event in a track and field meet.
   - **a.** Why is it important to coordinate running and jumping with this event?
   - **b.** What are some factors that might affect the number of steps or strides of a particular broad jumper?

STUDENT PAGE 14

2. Answers will vary; possible answers include: measure length of one step and divide this length per step into 50 m; or, walk 50 m and count your number of steps.

3. **a.** 50 m / 0.85 m per step = 58.8 steps or approximately 59 steps.

b. A description based on one step suggests a high potential for error. One of the main points of this lesson is to sense how a summary of this type is better determined by an finding an average.

c. One method would be to walk a distance of 50 m and count the number of steps.

4. Answers will vary on this problem. It may be necessary to set a different "standard" than 85 cm based on the results from volunteers.

a. This material was field tested with 9th and 10th graders. Using 85 cm as the standard, the taller students were clearly in one group. It might also be summarized by students that groups were determined by gender. However, at this age, males were generally taller in the field-test groups. Students 6 ft or taller were frequently in the group with steps measuring 85 cm or more.

b. Again, students shorter than 6 ft were in the group measuring 85 cm or less. Change the 85 cm to a value that more appropriately demonstrates the relationship to height given the students physical characteristics in your class.

2. How could you determine how many steps it takes you to walk a distance of 50 meters?

3. Jason wanted to know how many steps it would take him to walk a distance of 50 meters. He decided to estimate the length of one step by marking on the floor the starting and ending positions of his feet for one "typical" step. He measured this distance with a meter stick. Using this method, Jason described the length of his step as 85 centimeters. Use Jason's measure of 85 centimeters to answer the following:

a. Determine an estimate for the number of steps Jason would take to walk a distance of 50 meters.

b. Do you think your estimate in part a is accurate? Why or why not?

c. How could you evaluate the accuracy of this estimate?

4. Select a few volunteers from your class and using Jason's method, measure the length of one step of each volunteer.

a. Do any of your volunteers have a step measure greater than 85 centimeters? If yes, are there any additional descriptions or characteristics shared by these volunteers? Explain.

b. Do any of your volunteers have a step measure less than 85 centimeters? If yes, are there any additional descriptions or characteristics shared by these volunteers? Explain.

If Jason were to measure his step again, it is very likely that he would record a value different than 85 centimeters. Why? Because there are variations involved in a "typical step," an average value might be the best way to describe the number of steps needed to walk this distance.

Estimate Steps Required for Specific Distances

This investigation involves estimating the number of steps members of your class would take to walk a distance of 50 meters or any other designated distance. Consider the following procedure:

- Select two or three students from your class to carefully measure a distance of 50 meters (or designated distance) in a hallway of your school by using a measuring tape or meter stick.

5. Explanations will obviously vary, but it is unlikely the results will be the same for all five trials. The variable numbers can be explained by the problem with how to count the last step, the way a person walks (i.e., rigid and precision walking to a casual and uneven walking), mistakes in counting, uneven steps, etc.

6. Answers to the following problems depend on the collected data. By designing this problem around five trials, the likelihood for a mode is increased. Observing a mode, or a meaningful mode, however, is not important. Collecting this data and developing summaries is the main point of this part of the lesson.

   a. Use the collected data.

   b. Use the collected data.

   c. Use the collected data.

   d. Use the collected data.

7. Answers will again vary dependent on the collected data.

   a. Use the collected data.

   b. Use the collected data.

   c. Use the collected data.

STUDENT PAGE 15

- Mark a convenient starting and ending point on the floor of the hallway with masking tape.
- When given permission by your teacher, as many members of your class as possible should **walk** (do **NOT** run, jump, or skip for now) the measured distance of 50 meters (or designated distance) and count the number of steps it took each person to reach the end position.
- If you walked this distance, record your count as the result for Trial 1. Repeat this procedure by walking this same distance for four additional trials. For each trial, record the approximate number of steps counted. (Do not worry about any fractional step needed to reach the end position. Use your best judgment to estimate whether or not to count the last step. Record all trials as whole numbers.)

Your Name	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5

5. Did you expect to record the same number of steps for each of your trials? Why or why not?

6. Conduct a poll of the students in your class who walked the distance described.

   a. How many students recorded the same number of steps for all five trials?

   b. How many students recorded a different number for each of the five trials?

   c. How many students had four trials the same?

   d. How many students had two trials the same?

7. Determine the following summaries of the five trials you recorded.

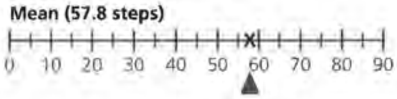
   a. the mean of the five trials

   b. the median of the trials

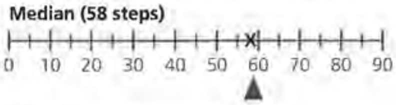
   c. the mode of the five trials (provided a mode value exists)

8. Answers will vary based on the collected data. Emphasize that each of the summaries represents a description of a physical characteristic of the student. The following is a representation of the values 56, 57, 58, 59, 59:

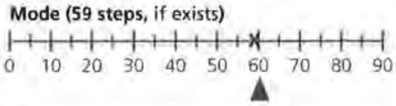
Mean (57.8 steps)



Median (58 steps)



Mode (59 steps, if exists)



9. Divide 50 m by the number of steps to summarize a student's walk of this distance. The final unit will be *meters per step*.

10. Answers will vary depending on the student's mean.

a. Divide the 50 m by the mean of the number of steps.

b. To determine the length in centimeters, multiply the meters per step by 100.

11. Answers will vary depending on the student's median.

a. Divide the 50 m by the median number of steps.

b. Again, multiply the meters per step by 100.

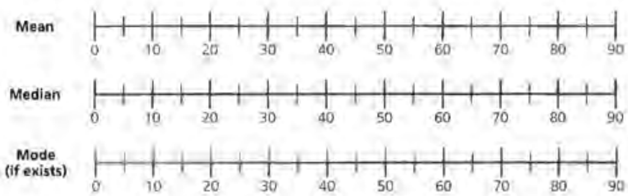
12. If there is a mode, use this value to complete parts a and b.

a. Divide the 50 m by the observed mode. (If a student has two modes, then there would be two answers to this part of the problem.)

b. Again, multiply the meters per step by 100 to obtain the number of centimeters per step.

STUDENT PAGE 16

8. Develop a visual comparison of your mean, median, and mode by recording each average on a number line similar to the following:



9. This experiment was based on counting the number of steps needed to walk approximately 50 meters. How could you use these results to determine the length of one of your steps?

10. Based on the **mean** value of the five trials, determine the length of one of your steps.

a. in meters.

b. in centimeters.

11. Based on the **median** value of the trials, determine the length of one of your steps.

a. in meters.

b. in centimeters.

12. Based on the **mode** value of the trials (provided your data set had a mode), determine the length of one of your steps.

a. in meters.

b. in centimeters.

13. Which average do you think is the best description of your "typical step"? Why?

14. What might be a reason for using each of the following in this particular example?

a. a mean as the best estimate

b. a median as the best estimate

c. a mode as the best estimate

13. Answers and explanations will vary. The field-test groups were able to describe appropriate reasons for selecting the mode, median, or mean based on their collected data. Which is the "best" is based on the recorded values. Ideally all of the descriptions are similar.

14. a. The mean is a good center value for data that is relatively similar. If the difference between the highest and the lowest recorded value is not great, this value is a good center and represents a good description of a "typical" step.

(14) b. The median (especially for a data set of 5) is a good center value. It is particularly useful when the difference between the highest or lowest is great. The median cancels out the extremes whereas the mean builds these values into the center.

c. Modes are easy to spot as good descriptions when it is between the highest and lowest values. The mode is not as good a description when it is a representation of either the highest or lowest value. Also, it is possible a mode does not exist for a particular student.

15. Answers will depend on the collected data from the class. It is not always clear why a student is selecting the specific value of a median, mode, or mean, but variation within the class provides a good comparison of students' physical characteristics.

16. Students should mark their recorded values from approximately 13 other students. This visual should suggest the connections discussed in the remaining problems in this lesson.

17. This problem again seems to suggest connecting steps with height, or possibly leg length or athletic development.

a. Results depend on the collected data.

b. In the field-test groups, this generally identified the taller students.

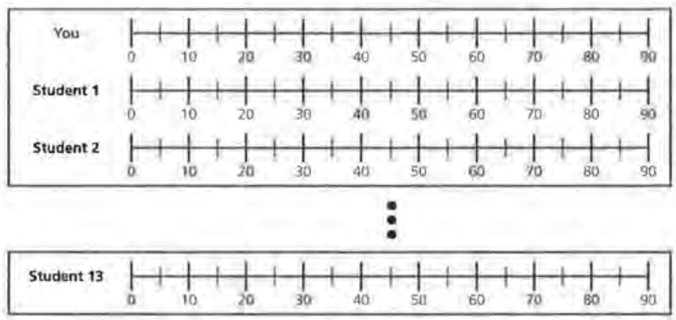
STUDENT PAGE 17

15. Select mean, median, or mode. Collect at least 13 averages of this type from students in your class. In other words, collect 13 means or 13 medians or 13 modes from the other students involved in this project. Do not mix the type of averages collected from the other members of your class! Copy and complete the following table:

Average selected in your sample: ____________________
(Mean, median, or mode)

Name	Number of Steps
⋮	⋮

16. Visually represent the results of the 13 summaries you collected by plotting the value for each student on the number lines provided on *Activity Sheet 2*. Include your summary in this collection.



17. Examine the data collected from your class.

a. Identify the two students in your data set who recorded the fewest number of steps.

b. Identify the two students in your data set who recorded the greatest number of steps.

STUDENT PAGE 18

(17) c. Height, leg length, forearm length, etc.

18. a. Months of birthdays, shoe size, and number of brothers and sisters showed no connections. Students generally did not suspect these characteristics but it provided a way to discuss connections. Number of sit-ups is a bit more complicated as it might be indirectly connected to other descriptions such as height, etc.

b. Height or forearm lengths were excellent selections to complete the rest of the problems. (However, some variation in developing these problems provides excellent discussions.) As a class, define the way a forearm, leg length, etc. is measured. For the field-test group, leg length was measured from the waist to the floor as a student stood straight.

c. Students complete the table for themselves and the 13 students selected from Problem 16.

c. As you consider the students identified in parts a and b, are there any descriptions or characteristics that distinguish the students who recorded the fewest number of steps from the students who recorded the greatest number of steps?

18. Consider the following additional descriptions or characteristics of the students in your class.

- Month of their birthdays (January = 1, February = 2, etc.)
- Height (in inches or centimeters)
- Length of their forearms (in inches or centimeters)
- Shoe size
- Number of brothers and sisters
- Circumference of wrists (in inches or centimeters)
- Number of sit-ups completed in 30 seconds

a. Which descriptions or characteristics in the list above do you think would not distinguish the students who recorded the fewest number of steps from the students who recorded the greatest number of steps?

b. Select one of the additional descriptions or characteristics you think might distinguish the two groups of students and explain why you selected this item.

c. Collect from the 13 students in your sample the value corresponding to the characteristic you selected in part b. Organize this additional piece of data in a chart similar to the one below.

Student	Additional Description (x-value)	Number of Steps (y-value)

19. Plotting the resulting points is *not* to be interpreted as a thorough treatment of correlation or other topics of linear regression. Simply use the points as a way to highlight a pattern or possible relationship of the two values. In the field-test groups, the general pattern was summarized by the observation that "as the heights of students increased, the number of steps needed to walk 50 m decreased."

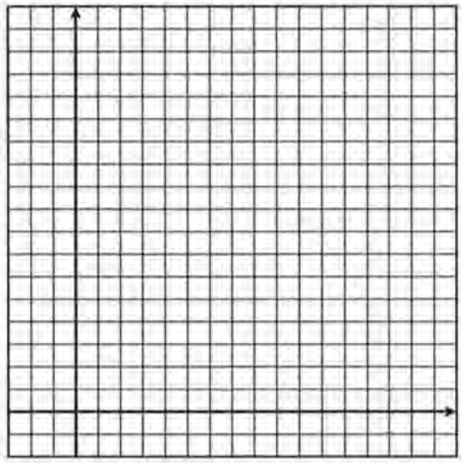
20. Although the answers depend on the descriptions selected, generally the graphs involving heights or forearms were connected as summarized in problem 19. Several of the other items did not indicate a pattern from the scatter plot.

The general pattern of the points suggests the connection. In the case of height or forearm length, a general line could be used to describe the connection. Although this could be developed into discussions of correlation, a general linear relation is a sufficient response to a suggestion of the relationship of the points. Details of correlation are left for other modules within this series.

21. Running the distance will change the results noticeably! In addition, if a student ran this distance for several trials, differences in the outcomes will be more obvious than the results for walking. This problem is not based on actually collecting the data by running the distance, but basically on a student's guess or conjecture. As a result, answers will vary. If possible, a small sample of students selected to run this distance might enrich this discussion. Develop this extension with a data sheet examining the number of steps involved in walking the distance and the number of steps involved in running the distance.

STUDENT PAGE 19

19. Using the x- and y-values as indicated in the chart, plot the points representing your sample of students on a coordinate grid similar to the one below. You may use *Activity Sheet 3*.



20. Do the points you plotted in problem 19 indicate a relationship between the Additional Description and the number of steps? If so, describe the relationship.

21. If you were allowed to run the 50-meter distance, how would the number of strides counted in this distance change?

SUMMARY

A value used to describe the length of a person's step or the number of steps needed for a person to walk a specific distance may best be described as a center of a data set. A mean, median, or mode can be used to identify the center.

When two variables are investigated, a coordinate grid may be used to visualize their relationship.

Practice and Applications

22. (Note: The distances involved are in yards and not meters. This changes the recorded values from those collected by the class.)

a. 42.833

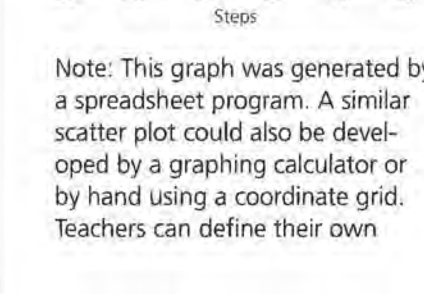
b. For values 41, 42, 42, 43, 44, and 45; the median is 42.5.

c. 42

23. Note: The chart involved in this problem organizes the data with the *x*-value as the number of steps. This is the reverse of what was set up in the Investigate section where the *y*-values represented the recorded step values. This is intentional as the goal is to think about a connection that is not involved in a cause-and-effect discussion. This is best achieved by making the roles of the *x*- or *y*-values interchangeable.

James Rockweger's height is highly suspect to error, especially given his recorded number of steps to walk the 30 yards.

24. The following graph excludes the recorded value for James Rockweger:



Note: This graph was generated by a spreadsheet program. A similar scatter plot could also be developed by a graphing calculator or by hand using a coordinate grid. Teachers can define their own requirements regarding the scales used for the *x*- and *y*-axes.

25. There does seem to be a pattern demonstrated with this data. In this case, the number of steps seems to be connected to the height, or "as the number of steps needed to walk 30 yards increased, the height of the students decreased."

26. Answers will vary, but an estimate of a line fitting the scatter plot would suggest James Rockweger's height to be 70 to 71 inches.

27. If you used 42 as his average (the mode value from his collected data set), then his height, suggested by the same line used in problem 26, is 68 to 70 inches.

STUDENT PAGE 20

Practice and Applications

22. Corey Reed collected the following data or number of steps taken as he walked a distance of 30 yards (note: this distance is measured in yards):

42, 41, 44, 43, 42, 45

a. Determine the mean of Corey's data set:

b. Determine the median of Corey's data set:

c. Determine the mode of Corey's data set:

23. Corey also collected the height and mean number of steps for several other students in his class. He recorded his collected data in the following chart:

Name	Number of steps (x)	Height in inches (y)
Sam Bland	37	74
Robert Rotella	38	76
Claud Salyards	42	68
James Rockweger	41	41
Devon Brooks	40	72
Anthony Balistreri	45	67
Rachel Jones	44	64
Jeffrey Durr	43	70
Laura Frye	42	69

Study the data set recorded by Corey. He might have erred in measuring! Corey most likely erred in measuring the height of which person? Explain why you think this measurement (or measurements) was incorrect.

24. Develop a scatter plot of the values indicated in the chart of problem 23 on a coordinate grid provided on *Activity Sheet 3*. DO NOT include any values from the data set you think represent an error in Corey's measurement.

25. Does height seem to be related to the mean number of steps recorded to walk 30 yards? Why or why not?

26. Estimate James Rockweger's height. How did you determine this estimate?

27. Estimate Corey's height. How did you determine this estimate?

UNIT II

Centers of Balance

LESSON 3

The Mean as a Center of Balance

Materials: rulers and raisins
Technology: graphing calculator
Pacing: 2 class sessions

Overview

This is the first of several lessons in which the *mean* is singled out as an especially important summary of a center. The first application of mean is directed at a location representing a center of balance. Balance is also expanded in other lessons of this module. The difficulty with this particular lesson is related to implementing the hands-on activities described and illustrated. Students are directed to place raisins along a centimeter ruler. Using the broad side of a pencil, students are asked to find a position along the ruler that balances the arrangement. The questions and problems related to the observations of the raisins and the ruler are not designed as precise experiments. Although raisins are described as "objects of *approximately* equal weight," the slight differences in the weights of raisins and in the rulers will contribute to different results from student to student. Even though precision is not emphasized, the process of developing a hands-on "feel" for a center of balance was highlighted as important by the field-test teachers.

Teaching Notes

This lesson is significant in setting up the work for lessons outlined later in this module. The role of a mean as a location along a number line providing balance is worked through several applications and examples throughout this module. Although students have worked with mean previously, this could be their first experience discovering the mean as outlined in this lesson. Closely related to this introduction of a balance point are the extensions of the mean to a weighted mean as presented in Lesson 4. Although possibly not obvious at this point in the module, this is an important lesson in setting the stage for the applications involved in population centers.

Organizing a hands-on component as outlined in the lesson is important and should be planned during the preparation of this lesson. (Logistics involved in providing the rulers and raisins to each student are important in estimating a time frame for this lesson.) This lesson can be introduced early in a geometry class since geometric references are minimal.

Follow-Up

Art, physics, and several other disciplines might also be incorporated in demonstrating to students a center of balance. Engineers involved in the design of aircraft, bowling pins, cars, bridges, and so on, are continually analyzing balance points. Information on the safety standards for cars, set up to rectify inappropriate centers of balance, are available and could be used to further the discussions presented in this lesson. These topics could be investigated by general searches on the Internet and/or consumer periodicals.

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LESSON 3

The Mean as a Center of Balance

In what situations is balance important?

Have you ever tried walking on a balance beam?

What could you do to improve your balance on the balance beam?

A keen sense of balance is often important in athletics. After watching the incredible movements of a gymnast on a balance beam, you may begin to appreciate the skills of balance necessary just to stay on the balance beam. This athlete's maneuvers, however, can also be interpreted from a science perspective. Studies of a gymnast's movements and sense of balance involve an intricate study of centers of balance.

OBJECTIVES

Determine the mean of a set of points presented on a number line.

Interpret the mean as a center of balance.

INVESTIGATE

Balance

A discussion of balance begins with a basic concept of distributing weights. To illustrate this concept, consider the following model: two raisins of equal weight are taped to the ends of a lightweight ruler. The goal is to balance the ruler with the attached raisins on the broad side of a pencil. (Note: if you attempt to develop this model, a lightweight, flat ruler will approximate the results developed in this section.)

Solution Key

Discussion and Practice

1. Possibly students will indicate the weight of the raisins, the type of ruler used in this model, the distances the pencil is located from the positions of the raisins, etc.

2. At this stage in the lesson, students are investigating. It is anticipated they will speculate the pencil should be between the two raisins (or the midpoint of the distance from raisin to raisin).

3. The weight of the raisins should not make a difference *if* the raisins weigh the same.

4. The balance point is primarily formed by the position of the raisins on the ruler, therefore, the midpoint of the distance from raisin to raisin is the best estimate of the balance point.

5. a.

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1. What variables affect the ability of the ruler to balance the raisins?

2. Move the broad side of the pencil underneath the ruler until a position is found that balances the ruler with the attached raisins. Where did you locate the pencil relative to the attached raisins to achieve a balance? Was this hard to do?

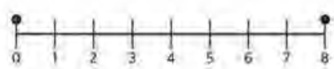
3. Would replacing the two raisins with two heavier objects (each weighing approximately the same) affect the balance? Try it by replacing the raisins with golf balls or similar objects.

4. Retape raisins to two other positions along the ruler. How would you change your estimate of where the pencil should be located to balance the ruler? Try and see if your estimate balances the ruler with the raisins.

The pencil represents a support used to balance the ruler. This support is called the ***fulcrum*** of the model. The challenge is to position the fulcrum (or support) so that the ruler will balance on the pencil.

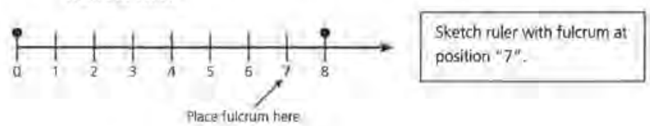
Center of Balance on a Number Line

Consider the ruler to be weightless. If that were possible, the weight of the raisins and the location of the raisins along the ruler determine the position of the fulcrum. The following number line can be used to represent the ruler and raisins:

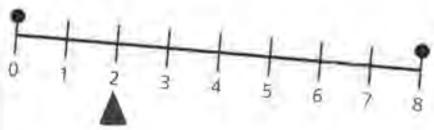


5. Consider taping raisins along the number line as illustrated. Place a fulcrum or support at various locations along the number line. Sketch the number line if:

   a. the fulcrum or support were placed at location 7 on this number line.



(5) b.



c.

6. Several descriptions could be given. The balance point P_1 could be considered the midpoint of AB. It could also be considered the mean of the values represented by A and B.

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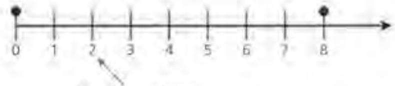
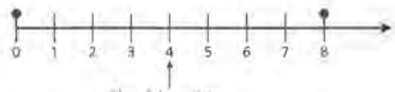
b. the fulcrum or support were placed at location 2 on this number line.



Sketch ruler with fulcrum at position "2"

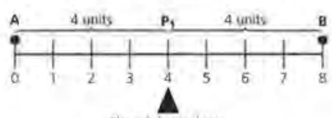
c. the fulcrum or support were placed at location 4.



Sketch ruler with fulcrum at position "4"

This last example balances the placement of the attached raisins. Similar to the situation observed when the pencil balanced the ruler with raisins located at two locations, placing a fulcrum in-between the weights as illustrated balances the arrangement of the weights.

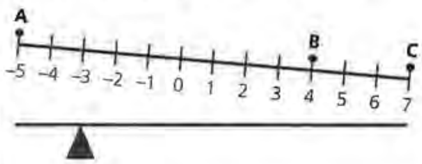
The distance from 0 to 4 is 4 units; similarly, the distance from 8 to 4 is 4 units. Therefore, the balance point is the location in which the distances to the right and left of the balance point are equal. Represent this balance point as P_1.



6. What descriptions of **center** could be used to summarize P_1?

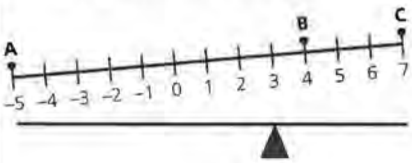
Recall, each point identified on the number line represents a raisin or "object of equal weight." Each object is also called a "**point mass**" when describing this type of problem. Suppose there are three raisins on an expanded number line located at A, B, and C as shown below.

7. At this point in the lesson, it is not expected the students will know the mean is the balance point. This problem (along with Problems 8 and 9) are developing that summary. For this problem, a placement of the fulcrum at any location that gives more "weight" to the points B and C is considered appropriate. As will be shown later, the balance point for this arrangement is 2. Therefore, placing the fulcrum at any location less than 2 will produce this type of imbalance.



This same effect will be observed for any value less than 2. Consider the answer appropriate if the fulcrum is placed at any location less than 2.

8. For the imbalance to produce the opposite effect, the fulcrum can be placed at any location greater than 2. Moving the fulcrum in that position gives the weight at location A the illustrated imbalance.



Again, there could be several locations of the fulcrum that could produce the effect illustrated above. Consider any of the locations closer to the weights at B and C as appropriate.

9. Answers will vary. The precision to calculate this balance point will be developed later in the lesson. Therefore, locations close to 2 should be considered appropriate.

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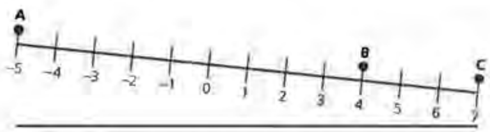
(Again, assume the number line is weightless. Obviously the ruler you used to demonstrate this model had weight and contributed to the location of the balance point. As the weight of the ruler is uniformly distributed, the weight and position of the raisins along the ruler essentially determine the balance point of each model.)

7. Where could the fulcrum be placed to produce the following sketch of the number line?

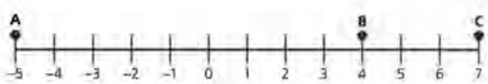


There could be several locations of the fulcrum that could produce the effect as illustrated by the above sketch. Copy the above sketch and shade the possible locations of the fulcrum that would result in the lack of balance as illustrated. Try modeling this sketch with a ruler and three raisins.

8. Where could the fulcrum be placed to produce the imbalance as illustrated in the following diagram?

Again, there could be several locations of the fulcrum that could produce the effect illustrated above. Copy the above sketch and shade the locations of the fulcrum that would result in the above imbalance of the number line. Also, experiment with a ruler and raisins to test your estimates.

9. Before you calculate the location of the fulcrum that would balance the three raisins, estimate where you would place the fulcrum to balance the model shown below.



10. a. 2 units

b. 5 units

c. 7 units

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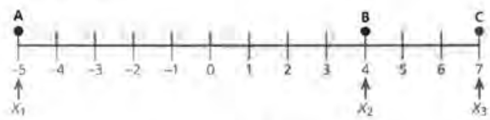
When two raisins were placed on the number line, the balance point was the location halfway between the weights. This location is also the ***mean*** or arithmetic average of the values representing the positions of the raisins. The mean centers the distances to the right and to the left of the location of the fulcrum. Can the mean similarly balance the distances for the example involving three points on the number line?

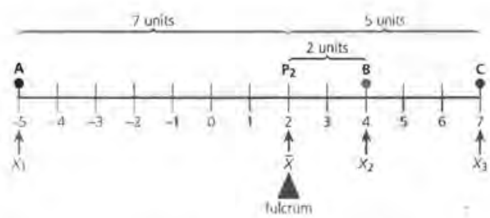
Let the value of each point on the number line be represented as x_1, x_2, and x_3. This arrangement is summarized by the following diagram.



Represent the position of the fulcrum for weights placed at points A, B, and C as P_2. What if the value of P_2 were estimated as $\bar{x}$, the mean of the values of the points? For this example, $\bar{x}$ would be:

$$\bar{x} = \frac{-5+4+7}{3} = 2$$

$$\bar{x} = 2$$



10. According to the diagram, how many units is

a. B from P_2?

b. C from P_2?

c. A from P_2?

11. a. points B and C

b. 7 units

c. A

d. 7 units

12. The total distances to the left of the fulcrum is equal to the total distances to the right of the fulcrum.

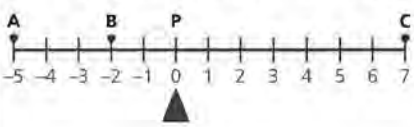
Practice and Applications

13.

14. The location of the center of balance is the mean of the points labeled in the above number line, or:

$$\frac{-5 + -2 + 7}{3} = \frac{0}{3} = 0$$

Therefore, the location of the balance point on the number line is 0:



(Verify with the students that the total distances to the right of the fulcrum equals the total distances to the left of the fulcrum.)

15. For each of the following items, students are expected to calculate the mean and construct a number line. If this problem is involved in class discussion, an excellent option would be to have the students construct a "before" and "after" number line. The "before" number line would be a visual estimate of the balance point before the calculation of the mean.

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11. Examine again the previous diagram involving three points on the number line.

a. What point or points representing raisins are located to the right of P_2?

b. Determine the total distance from P_2 to the point or points to the right of P_2.

c. What point or points are located to the left of P_2?

d. Determine the total distance from P_2 to the point or points to the left of P_2.

12. What characteristic of the mean balances the distribution of the raisins?

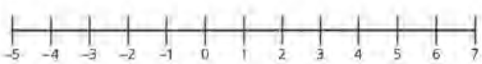
SUMMARY

Objects of ***equal weight*** positioned on a number line have a center of balance located at the ***mean*** of the locations of the objects. Placing a fulcrum at this center balances the total distances of the objects to the right of the fulcrum with the total distances to the left of the fulcrum. Balance is maintained by this equal distribution of the distances from the fulcrum.

Practice and Applications

13. Draw a number line as shown. Locate and label points A, B, and C. Let each point represent the position of an object of equal weight on the number line.

A = –5
B = –2
C = 7



14. Determine the location of the center of balance for the points specified in problem 13. Label the center of balance on the number line you designed as P.

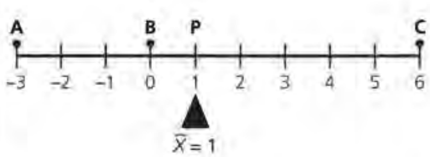
15. For each of the following, construct a number line and plot the points indicated. Determine the center of balance for each example (label the center of balance on the number line as point P).

a. A = –3, B = 0, and C = 6

b. A = –8, B = –7, and C = 6

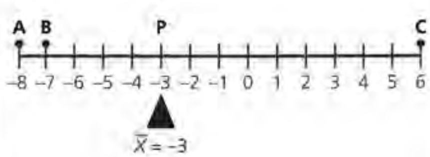
a. The balance point P would be located as the mean of the values –3, 0, and 6:

$$\bar{X} = \frac{-3 + 0 + 6}{3} = \frac{3}{3} = 1$$



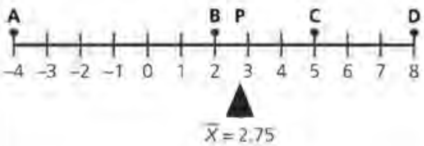
b. The balance point P would be located as the mean of the values –8, –7, and 6:

$$\bar{X} = \frac{-8 + -7 + 6}{3} = \frac{-9}{3} = -3$$



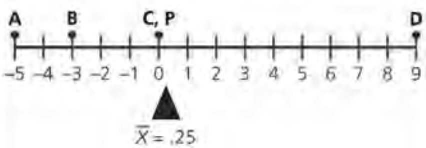
(15) c. The balance point P for this arrangement is the mean of the four points –4, 2, 5, and 8.

$$\bar{x} = \frac{-4+2+5+8}{4} = \frac{11}{4} = 2.75$$



d. The balance point P for this arrangement is the mean of the four points –5, –3, 0, and 9.

$$\bar{x} = \frac{-5+-3+0+9}{4} = \frac{1}{4} = .25$$



16. The process of calculating the position of B can be developed by setting up the mean. Let x represent the location of weight B along the number line.

$$\frac{-2+x}{2} = 4$$

$$-2 + x = 8$$

$$x = 10$$

Another approach in developing this problem is to note that the distance to the left of the mean is 4 – (–2) or 6. Therefore, the distance to the right of the mean should also be 6. This indicates that

x = 4 + 6 = 10.

17.

a. A rewrite of the mean using the illustration would be

$$\bar{x} = \frac{-5+-3+7+\boxed{x_4}}{4} = \boxed{1}.$$

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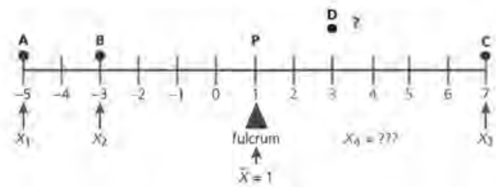
c. A = –4, B = 2, C = 5, and D = 8

d. A = –5, B = –3, C = 0, and D = 9

16. The location of the fulcrum P that balances two objects of equal weight placed at positions A and B is 4. You know A is located on the number line at position –2. Determine the location of point B. Explain how you determined your answer.

17. Four points are placed on a number line so that the center of balance is located at 1. The positions of 3 of the points A, B, and C are labeled on the following number line.



Use the following method to locate the fourth point, D.

a. Rewrite the following using a value or variable represented in the illustration to replace each "??".

$$\bar{x} = \frac{-5+-3+7+\boxed{??}}{4} = \boxed{??}$$

b. Solve the above for the location of point D or x_4 on the number line.

18. Consider the following two examples. Determine the missing values and plot the results on a number line for each.

Point A x_1 =	Point B x_2 =	Point C x_3 =	Point D x_4 =	Location of the Balance Point P $\bar{x}$ =
–5	–3	0	3	?
–4	2	?	4	1

b. Solve the above for the location of point D or x_4 on the number line.

$$\bar{x} = \frac{-5+-3+7+x_4}{4} = 1$$

$$-5 + -3 + 7 + x_4 = 4(1) = 4$$

$$-1 + x_4 = 4$$

$$x_4 = 5$$

18.

Point A x_1 =	Point B x_2 =	Point C x_3 =	Point D x_4 =	Location of the balance point P $\bar{x}$ =
–5	–3	0	3	$\frac{-5}{4}$ = –1.25
–4	2	2	4	1

19. Brian had a mean of 20.5 points on four 25-point quizzes in physics. He recalled his scores on three of the quizzes. They were 18, 24, and 22.

a. As the mean is less than two of the three known scores and only slightly higher than the third known score, the unknown score would need to be less than the mean score of 20.5. This could be visualized by placing the scores on a number line and discussing the distances to the left and to the right of the mean.

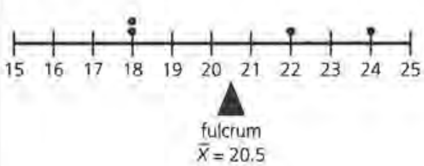
b. Let x_4 represent Brian's fourth quiz score. Therefore:

$$\bar{x} = \frac{18 + 24 + 22 + \boxed{x_4}}{4} = \boxed{20.5}$$

Solve for x_4 in the previous equation:

$$18 + 24 + 22 + x_4 = (4)(20.5)$$
$$64 + x_4 = 82$$
$$x_4 = 18$$

c.



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19. Brian had a mean of 20.5 points on four 25-point quizzes in physics. He recalled his scores on three of the quizzes. They were 18, 24, and 22.

a. Without actually calculating Brian's score on the fourth quiz, determine if this score is greater or less than his average of 20.5. Why?

b. Determine the specific value of Brian's fourth quiz score.

c. Represent Brian's scores and mean on a number line similar to the examples presented in this lesson.

LESSON 4

Weighted Averages

Materials: Unit II Quiz
Technology: graphing calculator
Pacing: 2 class sessions

Overview

This lesson introduces some rather familiar problems involving weighted averages. The goal is to connect each example to a center of balance on a number line as introduced in Lesson 3. This provides a new look to some familiar problems. Some students will find this connection interesting; other students might find the presentation of average as a center of balance more difficult.

The interpretation of a weighted mean as a center is used in several of the lessons of this module. The variety of problems is intended to demonstrate the importance of a center. The representation of a balance point as derived from a weighted mean is quickly expanded to two-dimensional models in the latter lessons of this module.

Teaching Notes

This lesson was considered challenging by the field-test teachers. The difficulty noted by teachers was not in the type of problems but in the new organization of the information. This lesson is designed to assist in illustrating several examples involving the location of weights (or raisins) on a number line. Connecting the representation of test scores to the raisins used in Lesson 3 (or, similarly, the number of two-point baskets in basketball to the number of raisins) is intended to emphasize the mean as a center of balance.

Technology

The use of the List options of many graphing calculators is growing in importance with this lesson. Although not necessary, it is possible to view the data presented in this lesson using the lists available on a graphing calculator. It is also possible to develop presentations of the problems through a spreadsheet application on a computer. Determining whether or not to consider these options should be based on the technology available and the timetable decided upon for this lesson. Each of these extensions would, of course, add to the preparation and the pace of the lessons.

Follow-Up

The problems presented in this lesson could easily have been expanded. Quiz or test scores, sports data, weights on a balance, and so forth, are available and could be used as further investigations of the lesson. A simple data collection representing the number of hours students in a class watched television the previous day, or the number of brothers or sisters each student has, or the number of books read during a specific period of time, could be used to determine a class average by the methods described in this lesson.

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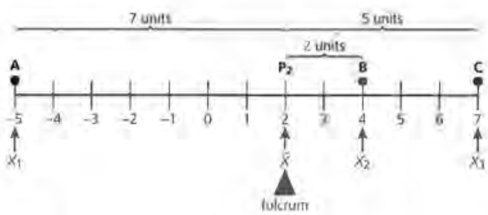
LESSON 4

Weighted Averages

What if the objects located along a number line are not of equal weight?

Suppose a weight at point A in the diagram below weighed more than the weights at points B and C. In what way would this change the location of a fulcrum representing the balance point?

How would you locate a balance point if the weights located along the number line were not of equal weight?



The number line illustrated in the previous lesson provided a visual way to represent the positions of objects of equal weights. The location of a balance point for these objects describes an important point, or center. If the number line is weightless and the objects located along the number line are of equal weight, then the "balance" point or center is where the total distances from the objects to the right of this center are "balanced" by the total distances from the objects to the left.

OBJECTIVES

Determine the center of balance by calculating the weighted mean of objects located along the number line.

Determine the weighted mean for problems of percents and similar applications.

INVESTIGATE

Balance Points

In the previous lesson, three raisins were located along the number line at the locations indicated as A, B, and C. Return

Solution Key

Discussion and Practice

1. In this problem, three locations are identified with raisins (each location is weighted with one raisin). Find the balance point with the original arrangement of the three raisins. As the middle raisin B is moved toward the raisin at C, the pencil also needs to move in that direction to maintain a balance.

2. a. The mean of the three locations of the raisins is

$$\bar{x} = \frac{-5 + 1 + 7}{3} = \frac{3}{3} = 1.$$

b. The mean of the three locations of the raisins is

$$\bar{x} = \frac{-5 + 6 + 7}{3} = \frac{8}{3} \approx 2.67.$$

Note: The movement of the fulcrum follows the movement of the middle raisin.

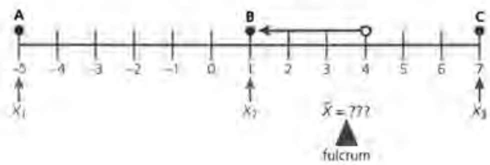
STUDENT PAGE 32

to your ruler and raisins. Tape three raisins along the lightweight ruler. Use the broad side of a pencil and find a position of the pencil that balances the ruler and the raisins. Identify one raisin as A, the middle raisin as B, and the third raisin as C.

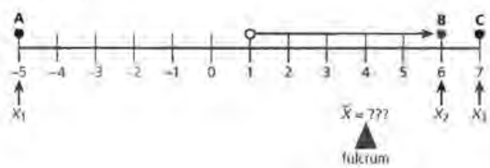
1. In which direction does the location of the fulcrum change as the raisin or weight identified at B is moved toward C? Try sliding a raisin represented in your model toward the raisin located at C. Now readjust the pencil representing the fulcrum. In what direction did you move the pencil to balance the new arrangement?

2. Consider each of the following changes to the locations of the raisins.

   a. Determine the new location of the fulcrum to balance the three raisins if the weight located at B is shifted to the position 1 on the number line.



   b. Determine the location of the fulcrum to balance the three raisins if the weight located at B is shifted to the position 6 on the number line.



3. **a.** The mean of the arrangement of the raisins in which B is right above C is

$$\bar{x} = \frac{-5+7+7}{3} = \frac{9}{3} = 3.$$

b. Students should observe that the illustrated change in the arrangement of the raisins shifts the balance point further to the right, therefore, students should agree with this statement. The movement of the raisin puts more weight on the right side. This in turn indicates the distance from the balance point of the raisins on the right is decreased.

4. **a.** This is essentially the same problem. Instead of three weights, the new arrangement has two weights, with the weight at location Q equal to the combined weights at B and C.

b. The following values for d_1 and d_2 can be read directly from the diagram:

d_1 = 8 units

d_2 = 4 units

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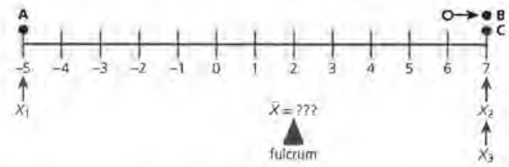
3. Continue shifting weight B to the right until it is located at position C as shown.



a. What is the mean of this arrangement of the three raisins?

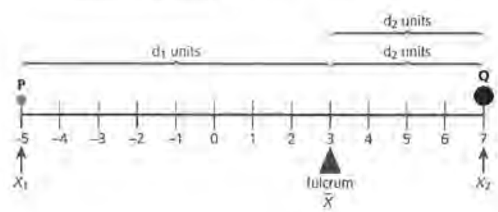
b. Do you agree or disagree with this statement: "The balance point of the 3 raisins shifts in the same direction as the shift of weight B." Why or why not?

Consider a weightless number line with two weights P and Q attached at the locations indicated. Consider the weight at location P to be equal to the weight of one standard raisin. Consider the weight located at Q to be equal to the weight of two standard raisins.

4. Consider the following.

a. Is this the same problem as presented in problem 3? Why or why not?

b. Determine the values of d_1 and d_2 by considering two raisins at location Q.



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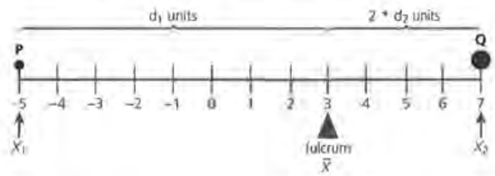
(4) c. Yes. The distance d_2 is weighted twice as location Q has two raisins attached to that position.

c. Are the total distances to the right of the fulcrum balanced by the total distances to the left?

Another way of dealing with this picture would be to take into account the combined weights of the raisins at location Q. As the weight represented at that location is twice the weight represented at location P, the distance from the balance point to Q is multiplied by 2. This is illustrated in the following diagram:



5. Organize the data from this number line by completing the following table:

Point	Weight W_i	Position on the number line x_i	Weighted value $W_i x_i$
P	____	–5	____
Q	2	____	14

6. Use the table developed in problem 5 to answer the following:

a. Why is x_i (position) multiplied by W_i (the weight of the raisin) located at that point?

b. What is the sum of the values calculated under the column heading "Weighted value" or "$W_i x_i$"?

c. What is the sum of the values under the column heading "Weight" or "W_i"?

d. What is the sum of $W_i x_i$ divided by the sum of W_i?

SUMMARY

The balance point of weights distributed along a number line is determined by the **weighted average.** The weight of each object is multiplied by its position on the number line to represent its weighted value. The sum of each value is then divided by the total weight of the objects to determine the weighted mean.

5.

Point	Number of raisins (unit of weight) W_i	Position on the number line x_i	Weighted value $W_i x_i$
P	1	–5	–5
Q	2	7	14

6. a. The distance is "weighted" by multiplying it by the number of units of weight (or, in this case, raisins).

b. $-5 + 14 = 9$

This represents the total sum of the weighted distances.

c. 3 or the number of units of weight (or, in this case, raisins).

d. The ratio is $\frac{9}{3} = 3$, the balance point.

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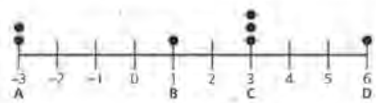
This is summarized by the following formula:

$$\bar{x} = \frac{\text{sum of } W_i x_i}{\text{sum of } W_i}$$

Consider the following weightless number line and objects of equal weight (for example, raisins) located on this number line.



7. Copy the above diagram.

a. Without actually calculating, label your estimate of the balance position as point E.

b. If a fulcrum is located at your estimate, determine the total number of weighted units to the right of the fulcrum.

c. Similarly, determine the total number of weighted units to the left of the fulcrum of your estimated point E.

d. Would you revise your estimate of point E in part a based on your answers to b and c?

8. Copy and complete the following chart for the diagram.

Point	Weight W_i	Position on number line x_i	Weighted value $W_i x_i$
A	_____	_____	_____
B	_____	_____	_____
C	_____	_____	_____
D	_____	_____	_____

9. Using the values determined from the chart in problem 8, calculate the location of the balance point by calculating the weighted mean. Locate and label this point as F on the diagram sketched in problem 7.

10. If the unit of weights were stated in grams instead of "the weight of a raisin," would the location of the balance point change? Why or why not?

---

7. a. Given what the students have learned so far, you would expect they would estimate the balance point close to the location of the three raisins piled together.

b. This answer depends on the specific guess of the balance point of the student. If, for example, the estimate was position 2, then the total weighted distances to the right would be

$3(1) + 1(4) = 3 + 4 = 7.$

c. Again, if the estimate was position 2, then the total weighted distances to the left would be

$1(1) + 2(5) = 1 + 10 = 11.$

d. The total distances to the left should equal the total distances to the right; so the fact that the above estimates do not equalize the distances indicates 2 is not the location of the balance point. An estimate of the balance point would need to be shifted to the left so as to increase the value of the total distances to the right and decrease the total distances to the left.

8.

Point	Number of raisins (unit of weight) W_i	Position on the number line x_i	Weighted value $W_i x_i$
A	2	-3	-6
B	1	1	1
C	3	3	9
D	1	6	6

9. $\bar{x} = \frac{\Sigma W_i x_i}{\Sigma W_i}$

$= \frac{-6 + 1 + 9 + 6}{2 + 1 + 3 + 1} = \frac{10}{7} \approx 1.43$

Note: It might be important to show how this value balances the distances to the right and to the left (an important summary about the mean). Using the above value of 1.43 which is rounded to 2 decimal places, the distances would be as shown on the next page.

Weighted distances to the right of the mean:

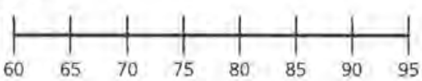
$1(6 - 1.43) + 3(3 - 1.43)$
$= 4.57 + 4.71$
$= 9.28.$

Weighted distances to the left of the mean:

$2(-3 - 1.43) + 1(1 - 1.43)$
$= -8.86 + -.43$
$= -9.29.$

10. Changing the units to grams does not change the location of the mean. As the mean represents the balance point, changing the unit of weight of the object does not change the balance point of the arrangement of objects.

11.



a. The values represented on this number line are percents. For example, the 60 represents 60%, the 70 represents 70%, etc.

b. Each ● represents the "weight" of one quiz. As the exams are described or weighted in terms of a quiz, each unit represents a quiz.

c. A score obtained on an exam is "four times as much as a quiz." Therefore, each exam is represented as the following:

d. The balance point represents the mean or average score (as a percent) of the exams and quizzes.

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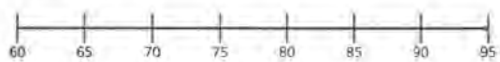
Changing Units on the Number Line

In the previous problems, the number line was a convenient representation of the locations of weights. Many examples of weighted averages involve a similar setup, except that information other than location and weight is involved. Consider the following example.

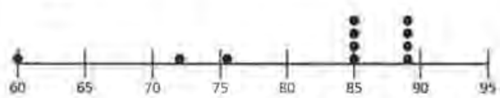
> Students enrolled in Ms. Clifford's math class frequently take either quizzes or exams. An exam is worth four times as much as a quiz. At the end of a grading period, Gail completed two exams and three quizzes. She got a score of 85% on the first exam, 89% on the second exam, and 72%, 60%, and 76% on the three quizzes. What is Gail's average for this grading period?

11. Before you begin to actually calculate Gail's average, consider a "picture" of this problem. A number line is sketched below. Copy it.



a. Instead of locations, what do the values indicated along this number line (60, 65, etc.) represent?

b. Consider locating ● along your number line. Instead of representing a unit of equal weight, what does a ● represent?



c. How is an exam represented in this picture?

d. What does the balance point of this diagram represent?

12. (The completed chart is at the bottom of the page.)

The mean is

$$\bar{x} = \frac{\Sigma W_i x_i}{\Sigma W_i}$$

$$= \frac{340 + 356 + 72 + 60 + 76}{4 + 4 + 1 + 1 + 1}$$

$$= \frac{904}{11}\text{, or about } 82.18\%.$$

13. The highest possible average for Steve would result from the highest possible score on his third quiz. If Steve obtained 100% on this quiz, then his mean score (as a percent) would be

$$\bar{x} = \frac{\Sigma W_i x_i}{\Sigma W_i}$$

$$= \frac{4(80) + 4(84) + 1(88) + 1(80) + 1(100)}{11}$$

$$= \frac{320 + 336 + 88 + 100}{11}$$

$$= \frac{924}{11} = 84\%.$$

14. The lowest average or mean would result from the lowest possible score on his third quiz. If Steve obtained 0% on this quiz, then his mean score (as a percent) would be

$$\bar{x} = \frac{\Sigma W_i x_i}{\Sigma W_i}$$

$$= \frac{4(80) + 4(84) + 1(88) + 1(80) + 1(0)}{11}$$

$$= \frac{320 + 336 + 88 + 80 + 0}{11}$$

$$= \frac{824}{11}\text{, or about } 74.91\%.$$

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12. Copy and complete the following chart.

Exam or Quizzes	Number of units of weight W_i	Score on the number line x_i	Weighted value $W_i x_i$
1st Exam	4	85%	____
2nd Exam	____	____	____
1st Quiz	____	____	____
2nd Quiz	____	____	____
3rd Quiz	____	____	____

Calculate the weighted mean of the scores.

13. Steve is also in Ms. Clifford's class. He has not yet completed the third quiz. He scored 80% on the first exam, 84% on the second exam, 88% on the first quiz, and 80% on the second quiz. Steve is hoping a good performance on the third quiz can pull his average for this grading period up to 85%. Is this possible? Find the highest possible average Steve could receive after he completes the third quiz.

14. What is the lowest average Steve could receive for this grading period?

Practice and Applications

15. Consider again the problem from Lesson 2 involving the number of steps students recorded to walk a distance of 50 meters. The following data was organized from Ms. Clifford's class.

Number of students	Number of steps to walk 50 yards
3	55
2	60
5	62
4	64
1	65

a. If everyone participated in this project, how many students are in Ms. Clifford's class?

b. Draw a number line with equal segments marked off to represent the units discussed in this problem. What units are represented on this number line?

(Chart, problem **12.**)

Exam or quizzes	Number of units of weight W_i	Score on the number line x_i	Weighted value $W_i x_i$
1st Exam	4	85%	340
2nd Exam	4	89%	356
1st quiz	1	72%	72
2nd quiz	1	60%	60
3rd quiz	1	76%	76

Practice and Applications

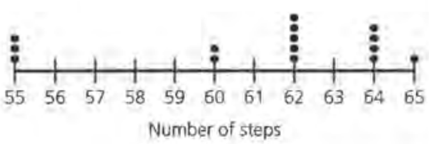
15.

Number of students	Number of steps to walk 50 yards
3	55
2	60
5	62
4	64
1	65

a. 15 students

b. The units marked on the number line indicate the number of steps students recorded in a 50-yard walk.

c.

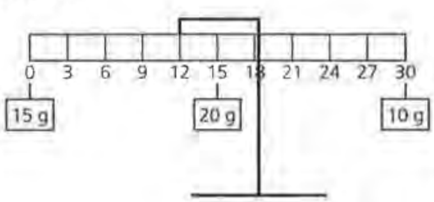


d. $\bar{x} = \dfrac{\Sigma W_i x_i}{\Sigma W_i}$

$= \dfrac{3(55) + 2(60) + 5(62) + 4(64) + 1(65)}{15}$

$= \dfrac{916}{15}$, or about 61.1

16. a.



b. The weights in grams replace the raisins in this problem.

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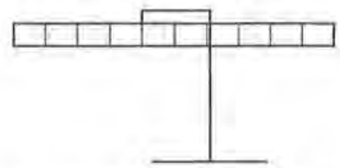
c. Complete the drawing by placing ● on the number line as indicated by this problem.

d. Determine the average number of steps this class takes to walk the 50 meters.

16. An apparatus was developed in a physics lab that investigates the location of a balance point. A relatively lightweight rod was calibrated in inches, starting with 0 and ending with 30 inches. A weight of 15 grams was suspended from the 0 location of the rod, 20 grams at the location 15 inches, and 10 grams at the location of 30 inches.



a. Complete a copy of this picture by showing where the weights would be suspended.

b. What replaces the raisins in this picture?

c. If the weight of the rod is not considered, at what location along this rod would a balance of the suspended weights be obtained?

17. There are 26 students in Mr. Landwehr's Geometry class: 12 boys and 14 girls. An exam was completed by each of the students. The boys averaged 73.2% and the girls averaged 80.5%.

a. If a number line were constructed to represent this problem, what scale or units would be represented on this number line?

b. Represent this problem on a number line.

c. If the symbol ● was used in this problem, what does each ● represent?

d. What was the overall class average for this exam?

18. In Ms. Mastromatteo's Geometry class, 12 girls average 82.1% on this exam. The overall class average for 22 students was 84.4%. What was the average for the 10 boys?

c.

$\bar{x} = \dfrac{\Sigma W_i x_i}{\Sigma W_i}$

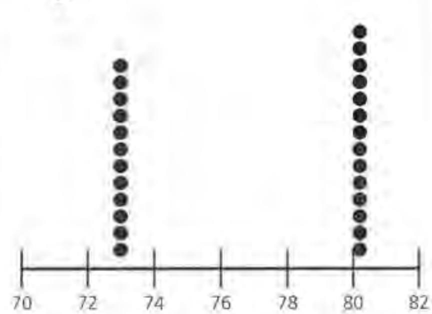
$= \dfrac{(15g)(0in) + (20g)(15in) + (10g)(30in)}{(15g + 20g + 10g)}$

$= \dfrac{600}{45} \approx 13.33$ inches

A fulcrum should be placed at the point labeled 13.33 inches on the rod.

17. a. The units along the number line would represent percents.

b.



c. Each ● represents one male student or one female student.

d. $\bar{x} = \dfrac{\Sigma W_i x_i}{\Sigma W_i}$

$= \dfrac{12(73.2\% + 14(80.5\%)}{26}$

$= \dfrac{878.4 + 1127}{26}$

$= \dfrac{2005.4}{26}$, or about 77.1%

18. Use a similar setup for this problem as was developed in previous problems. In this case, however, the overall class mean is known.

Let x represent the average percent score for the 10 boys; therefore,

$\bar{x} = \dfrac{\Sigma W_i x_i}{\Sigma W_i} = 84.4\%$

$= \dfrac{12(82.1\% + 10(x)}{22}$

$= 84.4(22) = 12(82.1) + 10x$

$= 1856.8 = 985.2 + 10x$

$= 10x = 871.6$

$= x = 87.16\%$

Extension

19. According to the chart, the three players attempted twenty 3-point shots and completed eight of the shots. If this represents the team totals, then $\frac{8}{20}$ or 40% is the team's percentage.

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Extension

19. Consider the following problem:

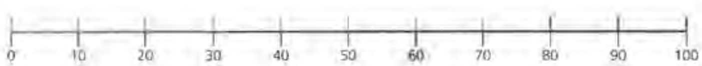
On February 11, 1993, the Phoenix Suns played the Golden State Warriors in an NBA basketball game. Phoenix won the game by a score of 122 to 100, largely because one of the Phoenix players, Danny Ainge, made several three-point shots. Here is a list of those players involved in three-point shots at this game:

Player	3-Point Shots Attempted	3-Point Shots Completed	Completed / Attempted	% Completed
Barkley	1	0	$\frac{0}{1}$	0.0%
Majerle	7	1	$\frac{1}{7}$	approximately 14.3%
Ainge	12	7	$\frac{7}{12}$	approximately 58.3%
Team Totals	____	____	____	____

Complete the above chart given the data for the three players.

The team's three-point percentage could be found by various methods. Consider the following setup to determine the team's three-point percentage.

20. This number line was started to illustrate the team's three-point average.



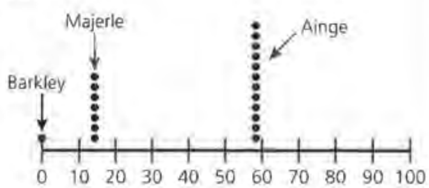
What do the units (i.e., 0, 10, 20, etc.) represent on this number line?

Player	3-Point shots attempted	3-Point shots completed	Completed / Attempted	% Completed
Barkley	1	0	$\frac{0}{1}$	0.0%
Majerle	7	1	$\frac{1}{7}$	approximately 14.3%
Ainge	12	7	$\frac{7}{12}$	approximately 58.3%
Team Totals	$\boxed{20}$	$\boxed{8}$	$\boxed{\frac{8}{20}}$	$\boxed{40\%}$

20. The units on the number line represent the percent completed of the 3-point shots.

21. Majerle is represented by the seven ●. The pile of seven represents the seven 3-point shots Majerle attempted. They are placed at the relative position of 14.3% on the number line to represent the percent completion he recorded for this game. Each ● represents an attempted shot by Majerle.

22.



23. $\bar{x} = \frac{\Sigma W_i x_i}{\Sigma W_i} = \frac{1(0) + 7(14.3) + 12(58.3)}{20}$

$= \frac{0 + 100.1 + 699.6}{20} = \frac{799.7}{20}$

$\approx 39.99\%$

24. According to chart in Problem 19, the team's average 3-point shooting percentage was 40%.

25. Yes, they are the same and should be the same (within round-off error). They are calculating the same average.

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21. Seven ● are placed on this number line representing one of the players involved in this problem. Who is represented by these seven units and what does each unit represent? Why are the ● symbols placed on the number line at a position between 10 and 20?

22. Copy and complete the above number line representing each of the players attempting a three-point shot for the Phoenix Suns.

23. Using the setup suggested by this diagram, find the team's average three-point shooting percentage.

24. Using the chart in problem 19, what was the team's average three-point shooting percentage?

25. Are the averages determined in problem 23 and problem 24 the same? Should they be?

UNIT III

"Raisin" Country

LESSON 5

Balancing a Point-Mass Triangle

Materials: heavy paper, raisins, tape, *Activity Sheet 4, Activity Sheet 7*
Technology: graphing calculator
Pacing: 2 class sessions

Overview

The major theme of this section continues to demonstrate *mean* as a center of balance. This lesson moves the number line and the previous one-dimensional representations of a balance point to the two-dimensional models. This lesson specifically focuses on a triangle and the location of a balance point represented by a triangle's vertices. The triangles presented on the blackline masters for this lesson are cut out of heavier paper; raisins are taped to the vertices of each triangle. This represents the *point-mass* triangles described in this lesson. Point-mass problems consider the weighted objects (or raisins) as the defining characteristic of the shape. The point at which each triangle model balances on the end of a pencil is described as the balance point of that triangle. This point is compared to a centroid, or the coordinate point represented by the mean of the coordinates of the vertices.

This lesson develops another method in locating a balance point (for a point-mass model) by a process described as "collapsing the raisins." This process is used to demonstrate several properties of a triangle and is used in several of the lessons involving other geometric shapes presented in the lessons following a student's work with triangles.

Teaching Notes

The topics in this lesson are reinforced by the hands-on development of the point-mass models. The triangle models will most dramatically indicate the connections of the balance point to the centroid and the collapsed point. For triangles, this point is also connected to the intersection of the medians. As the geometric figures expand to quadrilaterals and other polygons, the connection of the hands-on locations to the calculated points are not as apparent. This lesson, therefore, sets up important topics for several of the lessons involved in the remaining portions of this module. The development of the models to highlight the special properties and characteristics of triangles is important. If, however, it is not possible for every student to construct the paper-and-raisin models, a few of these models constructed by the teacher will be sufficient for solving the problems presented.

It is suggested the models be stored in a envelope for each student until the completion of Lesson 8 since each new lesson expands on the previous models.

Technology

This lesson could be used to introduce the LIST options of a graphing calculator. Specific directions for work with a graphing calculator are provided in Lesson 7 as the number of vertices and the number of raisins are expanded. A simple introduction using the triangle models is encouraged if this type of calculator work is new to the students. If a computer lab environment is available, you may want to represent the data as outlined in the lesson by using a spreadsheet application. Using either a graphing calculator or a spreadsheet (or both) is an excellent method to begin working with several new topics of center.

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LESSON 5

Balancing a Point-Mass Triangle

Do you think there is a balance point for three points located on a plane?

If a balance point exists, how can you find it? Is there a model similar to the ruler, raisins, and pencil?

When might it be important to know this balance point?

Balancing equal weights on a number line involves finding a point or center that evenly distributes the total distances of each weight to the left and to the right of the balance point. Expand this idea to three noncollinear points. Consider objects of equal weight located at three points on a sheet of paper.

OBJECTIVES

Determine the balance point of a point-mass triangle by experimentation.

Determine the centroid of a point-mass triangle.

Construct the intersection of the medians of a triangle.

Summarize the relationship of balance point, centroid, and intersection of medians.

INVESTIGATE

Balancing the Triangle

Triangles represent one of the most basic geometric shapes you will study. More complex geometric shapes are frequently investigated by dividing them into triangles! Triangles are described and classified by their angle measures and the lengths of their sides.

- A triangle in which each of its three angles has a measure less than 90 degrees is called an *acute triangle*.
- A triangle in which each all three sides are of equal length is called an *equilateral triangle*.

Solution Key

Discussion and Practice

1. The following descriptions are the most common in classifying and describing triangles:

Angle Descriptions

Acute Triangle: A triangle in which each of the three angles has a measure less than 90°.

Obtuse Triangle: A triangle in which the measure of one angle is greater than 90°.

Right Triangle: A triangle in which the measure of one angle is equal to 90°.

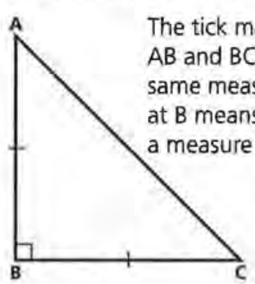
Side Descriptions

Scalene Triangle: A triangle in which the measures of all sides are different.

Isosceles Triangle: A triangle in which the measures of at least two sides are the same.

Equilateral Triangle: A triangle in which the measures of all three sides are the same.

2. The sketch developed by a student should combine the description of at least two sides of equal length and one angle equal to 90°. Many examples could be developed. Shown is one example.



The tick marks indicate AB and BC have the same measure. The box at B means angle B has a measure of 90°.

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Discussion and Practice

1. Research a geometry book and write a short summary of the angle descriptions and side descriptions for the following types of triangles.

Angle Descriptions	Side Descriptions
Acute Triangle: A triangle in which each of the three angles has a measure less than 90 degrees	**Scalene Triangle:**
Obtuse Triangle:	**Isosceles Triangle:**
Right Triangle:	**Equilateral Triangle:**

2. Sketch a diagram of a triangle classified as an ***isosceles right triangle.***

3. Look at the triangles on *Activity Sheet 4*. Select Triangle 1 and one other triangle. Determine the following measures of the triangles by using a protractor and ruler if necessary.

Triangle 1	Triangle ____ (your choice)
Length of P_1P_2: ____	Length of P_1P_2: ____
Length of P_2P_3: ____	Length of P_2P_3: ____
Length of P_1P_3: ____	Length of P_1P_3: ____
Measure of angle P_1: ____	Measure of angle P_1: ____
Measure of angle P_2: ____	Measure of angle P_2: ____
Measure of angle P_3: ____	Measure of angle P_3: ____

4. Refer to the solutions to problem 3.

a. Was it necessary to measure each angle with the protractor? Why or why not?

b. Are there any special characteristics of a triangle you used to determine the value of an angle or a side? If yes, describe the characteristics.

5. Using the approximate angle and side values summarized in problem 1, how would you describe:

a. Triangle 1

b. Triangle ____ (your choice)

3.

Triangle 1	Triangle ____ (Student's Choice)
Length of P_1P_2: 11.0 cm	Length of P_1P_2:
Length of P_2P_3: 10.8 cm	Length of P_2P_3:
Length of P_1P_3: 8.5 cm	Length of P_1P_3:
Measure of angle P_1: 65°	Measure of angle P_1:
Measure of angle P_2: 45°	Measure of angle P_2:
Measure of angle P_3: 70°	Measure of angle P_3:

4. Students might be able to summarize that the measures of a triangle add up to 180°. As a result, knowing two angles does not require measuring the third angle. Some students may also use some of the summaries of an isosceles triangle, namely, if two angles are equal, then the sides opposite the angles are congruent.

5. **a.** Triangle 1: Scalene, acute triangle. (This triangle is very close to an isosceles, acute triangle. This designation would depend on how round offs are handled.

   b. Results depend on the triangle selected by students and the estimated measures recorded by students.

6. Frequent reference is made to poster paper in this module. Poster paper is simply a heavier weight paper that will allow raisins to be taped and still maintain the shape of the figure. Most office supply stores stock paper of this type.

   a. Follow directions as indicated in the Student Edition.

   b. Follow directions.

   c. Follow directions. It is very important that the raisin be taped over the vertex.

   d. Throughout this module, directions indicate students should balance the figures with the raisins taped to each vertex. Why the raisins? There are actually two centers involved in the models; the first center is where the raisins or objects balance on a weightless plane (this is described as the balance point of a point-mass model). The second center involves the weight of the actual plane or poster paper. In most calculus applications, this uniform sheet is called a **lamina.** For most of the figures developed in this module, the locations of the two centers are almost the same. This module, however, is focusing on the balance point of the objects or raisins. Therefore, all models should have the raisins taped to the figures balanced. As the weight of the poster paper is uniformly distributed, the distribution of the raisins primarily determines the location of the observed balance points. It should continually be emphasized, however, that a balance point observed from each model is an approximation—variations of the calculated center and the balance point can be partially explained by the weight of the paper. (Ideally, the balance point of a point-mass problem is based on a plane with no weight.)

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6. In this investigation, you will use poster paper, raisins, and a pencil to investigate the balance point of a triangle. The triangle is outlined by poster paper. Raisins will be placed as objects of approximately equal weight at each vertex. Balancing this model finds a location that balances the weight and distribution of the poster paper and the raisins. The raisins represent a model called a ***point-mass*** distribution. The poster paper represents a model called a ***lamina.***

   a. Cut out Triangle 1 from *Activity Sheet 4.*

   b. Trace the shape of this triangle on a firm piece of poster paper, labeling each vertex of the triangle as indicated from the diagram. Cut out the triangle from the poster board.

   c. Tape one raisin at each vertex of the triangle (and as close to the vertex as possible).

   d. Balance this triangle on a pencil. You may want to use the eraser end of a new pencil. Warning! Don't give up too easily if the shape does not balance. Finding a balance point requires adjusting the pencil position in small increments. Keep in mind you are using the pencil as a fulcrum to locate a balance point similar to the point on a number line or ruler in the earlier lessons. (Suggestion: attempt to balance the figure with the raisins facing down. This helps stabilize the figure.)

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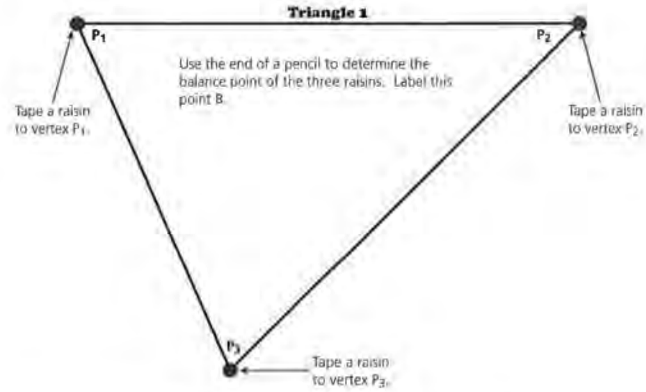
7. This point may be difficult for students to find. Work with the field-test group indicated some initial frustration with this process. Students were encouraged to move the pencil in small increments toward the part of the triangle that caused the model to fall off the pencil. The key is patience. If a student cannot find the point after a reasonable amount of time, assist the student or develop a class model and use the data from this model. A representative location of point B is illustrated in problem 11.

7. When you are able to balance the figure, mark the position on the triangle as point B for "balance point." See the following diagram.



The balance point B is the point that "centers" the distribution of the weights and locations of the raisins and the poster paper. The weight of the poster paper is uniformly distributed in this model. As a result, point B is primarily determined by the locations and weights of the raisins.

Calculating the Centroid

8. An activity sheet is provided showing the placement of Triangle 1 on the coordinate system developed in Problems 7–14. For students to determine the coordinate values of their balance point B, have the students press through the poster cut out of the triangle at point B. Place the triangle on a copy of this arrangement and mark point B. An illustration of this is provided in Problem 11, along with the placement of the centroid C.

Point	*x*-value	*y*-value
P_1	−6	5
P_2	7	5
P_3	−2	−4
B	0	2

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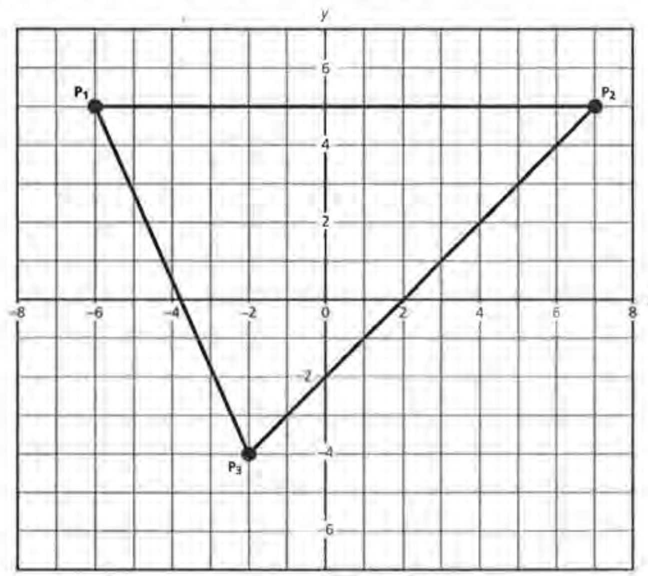
Calculating the Centroid

If you assume the weights of the raisins are equal, there is another method to determine B, the balance point. It is found by placing the triangle on an *xy*-coordinate system.



8. Estimate the coordinates of point B from the above placement of Triangle 1. Complete the following table based on this particular placement:

Point	*x*-value	*y*-value
P_1	____	____
P_2	____	____
P_3	____	____
B	____	____

9. $\bar{x} = \frac{-6 + 7 + -2}{3} = \frac{-1}{3} \approx -.333$

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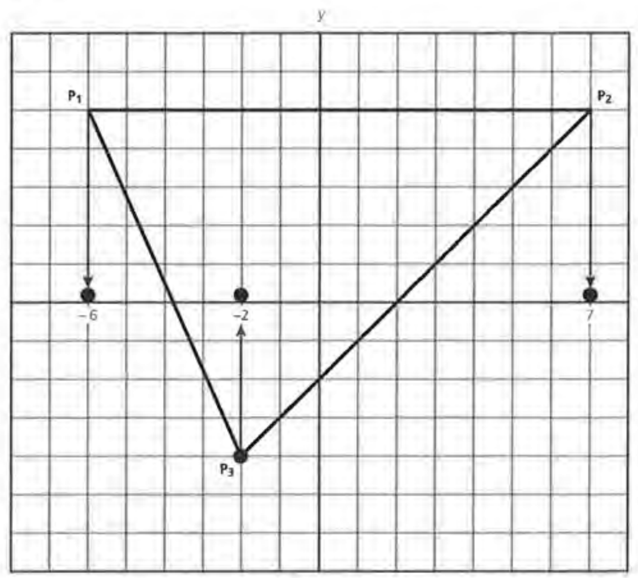
The balance point for this figure requires calculating both an x-value and a y-value. The horizontal component (or x-value) of this point can be found by imagining the raisins along the x-axis.



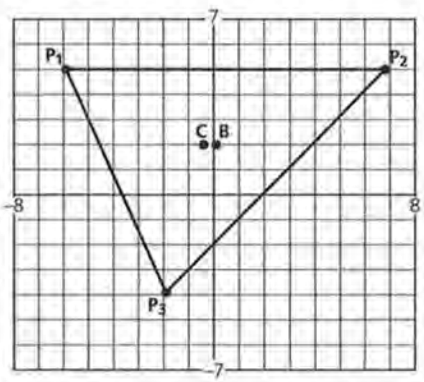
Recall from your work in Lesson 3 that the mean of points on a number line determined the balance point for weights of "equal" value. This was summarized as:

$$\bar{x} = \frac{(x_1 + x_2 + x_3)}{3}$$

9. Find the value of $\bar{x}$ using Triangle 1 as pictured above.

10. $\bar{y} = \frac{5+5+-4}{3} = \frac{6}{3} = 2$

11. The centroid would be C (–0.333,2). This point should be located on the same copy of this arrangement as the estimate of the balance point B determined in Problem 10. An example is provided. The location of B will vary as this estimate is dependent on the students' development of the model.



12. Points B and C are very close in the example provided in problem 11. The distance between the points is approximately 2 m. Examples from the field-test group indicated similar results. Expect distances between the two points to range from 0 to 1 cm in length.

13. Variations could be explained by the weight of the poster paper (see earlier comments), unequal weight of the raisins, the actual placement of the raisins, the actual determination of the balance point, etc. (Some students might tape the raisins slightly off the vertex. They should be instructed to tape the raisins right on top of each vertex for the best results. Also, the eraser end of the pencil is a fair-sized "circle." Estimating a point from the location of balance leaves room for an estimation that changes the location of B.) Students might also point out that the coordinate values of C are also based on estimated values of points P_1, P_2, and P_3. As a result, C is also an estimation.

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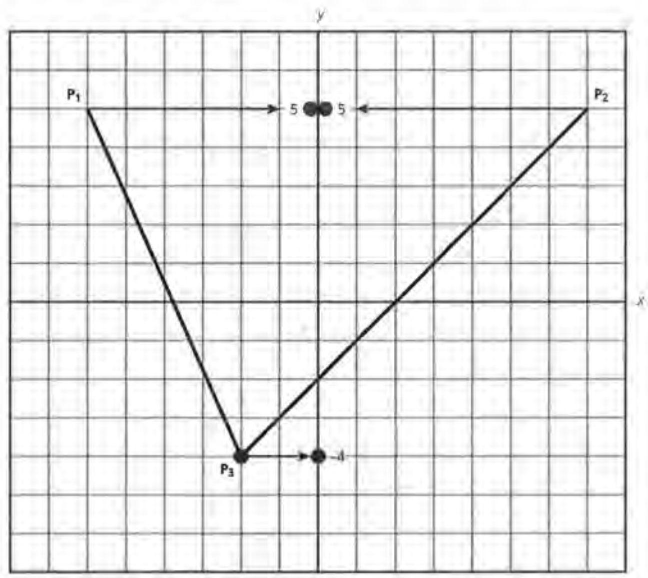
A similar approach can be taken to determine the vertical component (*y*-value) of the balance point.



The *y*-value of the balance point would be determined by the mean of the positions of the raisins along the *y*-axis. This is summarized as:

$$\bar{y} = \frac{(y_1 + y_2 + y_3)}{3}$$

10. Determine the value of $\bar{y}$ using Triangle 1 as pictured above.

11. The balance point calculated in this way is called the ***centroid.*** It is represented by C $(\bar{x}, \bar{y})$. Use the values calculated in problems 9 and 10 to locate the centroid of this particular triangle. (Label this point as C.)

12. How does the location of C compare to your location of B?

13. The balance point B is an approximation of the centroid C. What could cause the balance point obtained by experimentation to be different than the centroid?

14. a. (–6 – –.333) + (7 – –.333) + (–2 – –.333) = (–6 + .333) + (7 + .333) + (–2 + .333) = (–5.67 + 7.33 – 1.67) = –.01 or 0

Point out to students the sum is not exactly 0 due to an approximation of $\bar{x}$.

b. (5 – 2) + (5 – 2) + (–4 – 2) = 3 + 3 + –6 = 0

Working with the Medians

15. a. A relatively careful measurement of the midpoints and construction of the medians will result in the medians intersecting in a point.

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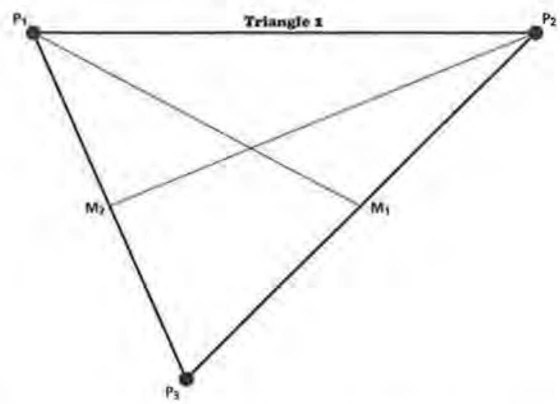
14. The centroid "balances" the vertical and horizontal distances of the raisins on the plane. Based on the placement of Triangle 1 in the previous diagrams, determine the following:

a. $(x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) =$

b. $(y_1 - \bar{y}) + (y_2 - \bar{y}) + (y_3 - \bar{y}) =$

Working with the Medians

Several well-known and important theorems in geometry point out the special features of the ***medians*** of a triangle. A median of a triangle is a line segment that connects a vertex to the ***midpoint*** of the side opposite that vertex. In Triangle 1, $\overline{P_1M_1}$ and $\overline{P_2M_2}$ are examples of medians:



An important theorem in geometry states ***"The three medians of a triangle intersect at one point."*** If you have not previously studied this theorem, you might want to experiment by constructing the three medians for each of the models on *Activity Sheet 4*. Make certain you verify the accuracy of this important theorem.

15. Return to the triangle you cut out of the poster board. Carefully measure the length of each side to determine its midpoint. Label midpoints as M_1, M_2, and M_3. Connect each midpoint to the opposite vertex.

(15)b. The intersection of the medians M is very close to the locations of C and B. This should be clear as the students add point M to the cut-out model.

c. "The medians of a triangle intersect *at the centroid of a triangle.*" (Or, "... the balance point of the triangle formed by three objects on a plane.")

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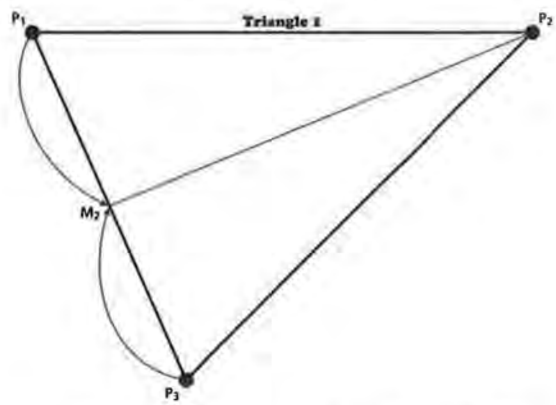
a. Do the three medians intersect at a point? If yes, label this point as M. (If your medians do not intersect, carefully remeasure the midpoints of the triangle. Remember the theorem!)

b. Compare M to locations of centroid C and the balance point B. Are the locations similar?

c. Complete the following statement based on the locations of M, C, and B: "The medians of a triangle intersect at ________."

Another important summary can be illustrated with the original arrangement of raisins taped to P_1, P_2, and P_3 of **Triangle 1.** Consider $\overline{P_1P_3}$ of the triangle formed with one raisin located at vertex P_1 and one raisin located at vertex P_3. (Again, consider each raisin to be of equal weight.)



The two raisins taped to the endpoints of this segment can be moved to M_2 or the midpoint of $\overline{P_1P_3}$. This is called ***collapsing the raisins.***

Remove the raisins from P_1 and P_3. Combine the two raisins at location M_2. The balance point B of the original triangle is also the balance point of the raisins arranged along segment $\overline{M_2P_2}$.

16. a. The balance point is maintained as the rearranged locations of the raisins does not change the equal distances to the left and to the right of the balance point of the three raisins. Students might see how this new location (M_2) is an average of the distances from P_1 and P_3. As an average, the *total* distances to the right and to the left are not altered. However, if this average of the distances is not observed, the following question b is developed to illustrate this.

b. $\bar{x}$ is $\frac{-4 + -4 + 7}{3} = \frac{-1}{3} \approx -.333$

and $\bar{y}$ is $\frac{.5 + .5 + 5}{3} = \frac{6}{3} = 2.$

The new balance point $(\bar{x}, \bar{y})$ for the arrangement of raisins along $\overline{M_2P_2}$ is (–.333, 2) or the same as the old balance point and centroid.

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16. Moving the two raisins to M_2 changes the arrangement of the three raisins but not the balance point B.

a. Why do you think B remains unchanged?

b. Review the following formulas for determining the midpoint:

$$\frac{x_1 + x_2}{2} \text{ and } \frac{y_1 + y_2}{2}$$

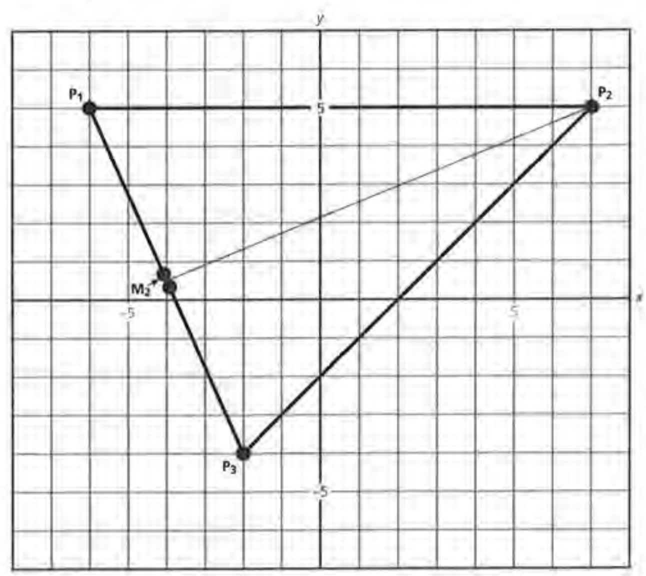
As M_2 is the midpoint of P_1 and P_3, the x-coordinate value of M_2 is:

$$\frac{-6 + -2}{2} = \frac{-8}{2} = -4$$

and the y-coordinate value of M_2 is:

$$\frac{5 + -4}{2} = \frac{1}{2} = 0.5$$

This location of M_2 can be verified by the previous placement of the triangle in the coordinate grid. Assume each tick mark is one unit.



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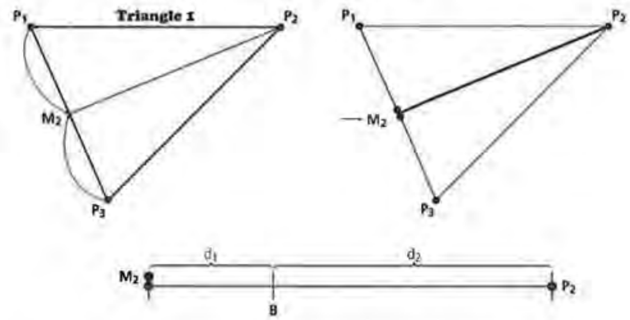
Determine $\bar{x}$ based on the placement of 2 raisins at M_2 and 1 raisin at P_2. Similarly, determine $\bar{y}$.

Is this balance point $(\bar{x}, \bar{y})$ the same as point B?

Return to the rearrangement of raisins by collapsing the 2 raisins to M_2 as illustrated in the following diagram:



Examine your triangle. Point B, discovered earlier in this lesson, should also be part of the segment $\overline{P_2M_2}$. Explain.

There exists a special relationship between d_1 and d_2 as illustrated above. Recall from Lesson 4 that the total distances for each unit of weight (or raisin) on one side of the balance point equals the total distances for each unit of weight on the other side. In this example, the resulting balance contributed by the two raisins at point M_2 and the one raisin at point P_2 is:

$$2d_1 = 1d_2$$

Also observe that:

$$d_1 + d_2 = M_2P_2$$

Therefore, by substitution,

$$d_1 + 2d_1 = M_2P_2$$

$$3d_1 = M_2P_2$$

$$d_1 = \frac{1}{3} M_2P_2 \text{ and } d_2 = \frac{2}{3} M_2P_2$$

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17. The above work indicates the location of the balance point is $\frac{1}{3}$ of the distance from M_2 and $\frac{2}{3}$ of the distance from P_2. What does this indicate about the intersection of the medians M or the location of the triangle's centroid C?

18. Suppose that instead of collapsing the raisins at P_1 and P_2 you collapsed the raisins located at vertex P_2 and at vertex P_3 to the midpoint M_1. Then $\overline{P_1M_1}$ would also be a median of the triangle. Copy the following representation of segment $\overline{P_1M_1}$ and the attached raisins.

a. Estimate on your copy of this segment the location of the **balance point** of $\overline{P_1M_1}$.

b. How is the balance point you estimated along $\overline{P_1M_1}$ related to the balance point B of the triangle?

c. When you collapse raisins, you are changing the arrangement of the raisins or weights but you are not changing the location of the balance point. Why is B (or C) also part of $\overline{P_1M_1}$?

19. According the theorem mentioned in this lesson, the three medians of a triangle intersect in one point. If the location of the intersection of the medians was based on the location you estimated for B, how would you explain that this point is also the intersection of the medians?

SUMMARY

Three raisins taped to a sheet of poster paper form a model of a point-mass triangle. The location of the **centroid** or point that balances the raisins can be estimated by:

a. using a pencil as a fulcrum to locate a point that balances the model

b. determining the means of the x- and y-coordinates of each of the raisins

c. locating the intersection of the medians

Practice and Applications

20. Cut out the second triangle you selected from the *Activity Sheet 4* on poster paper. Tape a raisin at each vertex. Using

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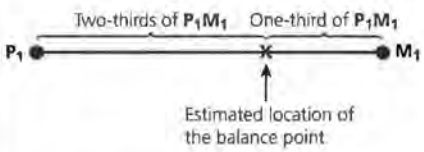
17. Students should observe in the model developed that B (or C) is located on or near the intersection of the medians. As the medians intersect at the balance point, this indicates the centroid is located along a median $\frac{2}{3}$ of the distance from the vertex to the midpoint of the opposite side.

18. a.



b. The balance point estimated on segment P_1M_1 is the same balance point B of the entire triangle.

c. Here again, collapsing the raisins kept the distances to the right and left equal.

19. One way to explain this is that there exists only one estimated balance point for the triangle. This one, unique point is the same point that balances any of the collapsed examples. For each example, the three raisins collapsed to a segment that fits the definition of a median, therefore, the balance point of the three raisins outlining the triangle is located along a median of the triangle. This point must be what *each* median has in common, namely, the point representing the intersection of the medians.

Practice and Applications

20. This problem duplicates the process developed for Triangle 1 with one of the other triangles provided.

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a pencil, find the point that balances the triangle. Label this as point B on your triangle.

21. Place your cut-out triangle on a coordinate system. Determine the *x*- and *y*-coordinates for each vertex and the balance point B. Copy and complete the following table for the triangle you selected:

Point	*x*-value	*y*-value
P_1	_____	_____
P_2	_____	_____
P_3	_____	_____
B	_____	_____

22. Determine $\bar{x}$ and $\bar{y}$ for the vertices.

23. Locate $(\bar{x}, \bar{y})$ as point C on the cut-out form of your triangle. (This represents the centroid of the triangle.)

24. Consider the following:

a. If the point that balances the triangle with your pencil is the centroid for a point-mass triangle, what would you expect about the points B and C?

b. Why might the specific locations of points B and C not turn out as expected?

25. Measure and mark the midpoints of each of the sides of your triangle. Label the midpoint of $\overline{P_2P_3}$ as M_1, $\overline{P_1P_3}$ as M_2, and $\overline{P_1P_2}$ as M_3. Connect each midpoint to the opposite vertex of the triangle. Do the medians you constructed intersect at one point? If yes, mark the point of intersection as M.

26. Using your triangle, measure the listed segments with a ruler and calculate the ratios:

$P_1M =$	$MM_1 =$	$\frac{P_1M}{MM_1} =$
$P_2M =$	$MM_2 =$	$\frac{P_2M}{MM_2} =$
$P_3M =$	$MM_3 =$	$\frac{P_3M}{MM_3} =$

27. Summarize the ratios found in the table above.

21. Answers will vary dependent on students' arrangement of the cut-out triangle on the coordinate grid. Emphasize that the placement of the triangle is not important. In other words, each student will have different coordinate values but the final location of the centroid within the triangle should be the same.

22. Results depend on the triangle selected and the estimates of the coordinate values for each vertex.

23. The location of C should be close to the student's estimate of the balance point B. This is similar to the process developed in Problems 14–19.

24. a. Students should conjecture that the points are the same.

b. Differences in the points are again related to varying weights in the raisins, estimating the center point from the pencil, specific placement of the taped raisins at the vertices, etc.

25. The medians should intersect in one point (or very close to one point). If they do not, students should repeat the process.

26. The following represent the results for Triangle 1. Obviously values are dependent on the triangle selected by a student, however, the final ratios should be close to 2.

$P_1M = 5.4$ cm	$MM_1 = 2.7$ cm	$\frac{P_1M}{MM_1} = 2.0$
$P_2M = 6.7$ cm	$MM_2 = 3.5$ cm	$\frac{P_2M}{MM_2} = 1.9$
$P_3M = 5.2$ cm	$MM_3 = 2.7$ cm	$\frac{P_3M}{MM_3} = 1.9$

27. A summary should be based on the fact that the longer segment is approximately $\frac{2}{3}$ the length of the entire segment; similarly, the shorter segment should be approximately $\frac{1}{3}$ the length of the entire segment. The ratio formed by these relationships is 2.

LESSON 6

Investigating Quadrilaterals

Materials: heavy paper, raisins, tape, *Activity Sheet 5, Activity Sheet 7*
Technology: graphing calculator or computer with spreadsheet software
Pacing: 2 class sessions

Overview

This lesson expands the work with triangles to the next level. Students will work with similar models involving quadrilaterals. Similar to the activities of the previous lesson, students will determine the balance point of a point-mass quadrilateral using a pencil as a fulcrum. They will then compare the location of this point to the location determined by the centroid and the collapsing process previously introduced. The collapsing process is used to highlight several of the special characteristics of the quadrilateral models.

Teaching Notes

The concerns and organization of this lesson are similar to those noted in Lesson 5. The models developed by the students bring together the balance point and the centroid as a location balancing distance and weight.

The placement of this lesson within a geometry class is important as some familiarity with quadrilaterals would be helpful. A few familiar geometric theorems are addressed in the lesson and supported with a discussion of center. Previous work with the theorems would decrease the time needed to complete this lesson. This lesson could be used, however, to highlight the theorems and topics used to further understand the special characteristics of a quadrilateral.

Technology

The use of the graphing calculator is suggested for this unit. In particular, the LIST options or the equivalent work with a spreadsheet application should be considered. Lesson 7 involves specific work with these topics; however, this lesson could be used to begin working with these applications.

Solution Key

Discussion and Practice

1.

Measure of angle	Length of side
$P_1 = 105°$	$P_1P_2 = 9.5$ cm
$P_2 = 75°$	$P_2P_3 = 6.3$ cm
$P_3 = 105°$	$P_3P_4 = 9.5$ cm
$P_4 = 75°$	$P_4P_1 = 6.3$ cm

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LESSON 6

Investigating Quadrilaterals

Which do you think is more stable, a 3-legged stool or a 4-legged stool? Why?

Imagine a quadrilateral with a raisin taped to each of the vertices. Assume each raisin weighs the same. Do you think there is one point on the plane of the quadrilateral designating the location of a fulcrum that would balance the four raisins? Why or why not?

OBJECTIVES

Determine the balance point of objects forming a quadrilateral on a plane through experimentation and coordinate geometry.

Identify special characteristics of a parallelogram by locating the balance point of a point-mass model.

Describe concavity by the location of a balance point.

Estimate the balance point of a point-mass model by collapsing raisins.

An extension of the investigation involving triangles is to attach additional objects of equal weight to the plane. How does a fourth object on the plane change the location of a balance point of the weighted objects?

INVESTIGATE

Parallelograms

Locating a fourth object on the plane outlines another familiar shape called a ***quadrilateral***. A quadrilateral is defined as a four-sided, closed-plane figure. Squares, rectangles, parallelograms, and trapezoids are a few special subsets of the larger set of quadrilaterals.

A good starting point is to examine a ***parallelogram***. Use Quadrilateral 1 from *Activity Sheet 5* to investigate the following problems.

Discussion and Practice

1. Quadrilateral 1 is a parallelogram. Using a protractor and a ruler, find and record the measures of the sides and angles of Quadrilateral 1.

2. **a.** Several observations could be made. In most cases, the observations are a result of the special characteristics of a parallelogram.

   - opposite sides are equal
   - corresponding angles are supplementary
   - opposite angles are equal

   b. Each of the above characteristics are true of all parallelograms.

3. **a.** Similar to the work with triangles, specific locations of B will vary. However, an example of a carefully constructed model is illustrated in problem 6.

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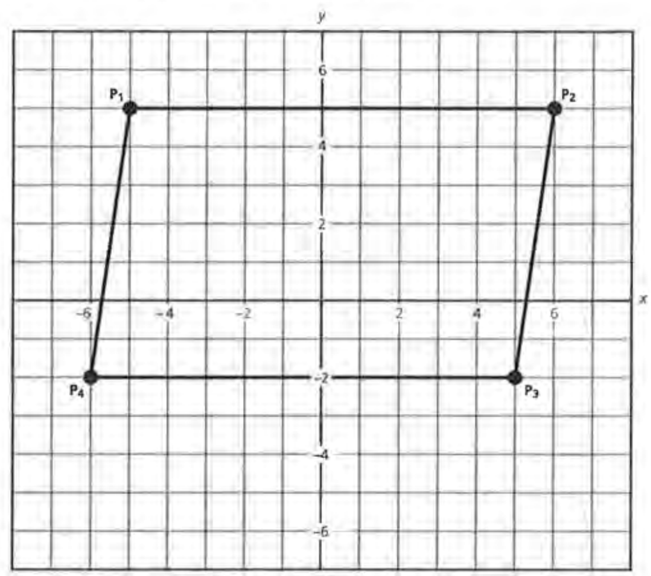
Measure of Angle		Length	
P_1 =	______	P_1P_2 =	______
P_2 =	______	P_2P_3 =	______
P_3 =	______	P_3P_4 =	______
P_4 =	______	P_4P_1 =	______

2. Examine your recorded measurements for Quadrilateral 1.
   a. Describe at least three special characteristics of this parallelogram.
   b. Which characteristics do you think are true of all parallelograms?
3. Tape raisins to each vertex of the quadrilateral.



   a. Using a pencil as a fulcrum, estimate the location of the balance point of the four raisins. Mark and label your estimate on the cut-out figure as point B.

(3)b. Answers will vary. Most students in the field-test groups anticipated the balance point to be located at the intersection of the diagonals. If this is cited as the basis of the estimated point, then emphasize how several of the following problems in this lesson will build on that idea! This is a special characteristic of parallelograms that will be reinforced through this investigation.

4.

Point	x-value	y-value
P_1	–5	5
P_2	6	5
P_3	4	–2
P_4	–7	–2

a. $\bar{x} = \frac{-5+6+4+-7}{4} = \frac{-2}{4} = -0.5$

$\bar{y} = \frac{5+5+-2+-2}{4} = \frac{6}{4} = 1.5$

b. See the diagram following this problem.

c. Based on the diagram, points B and C are approximately the same.

d. This problem is intended to get students to think about what a "median" of a rectangle would be if it existed. Based on the definition presented for a triangle, a median of a rectangle does not make sense. This emphasizes the special characteristics of a triangle and the medians of a triangle.

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b. Was B where you expected it to be located? Why or why not?

Recall that the calculation of the centroid C ($\bar{x}, \bar{y}$) could be used to locate the balance point of a triangle; $\bar{x}$ represents the mean of the x values of the vertices and $\bar{y}$ represents the mean of the y values of the vertices.

4. Copy and complete the following table based on the placement of **Quadrilateral 1** in the xy-coordinate grid as previously illustrated.

Point	x-value	y-value
P_1	____	____
P_2	____	____
P_3	____	____
P_4	____	____

a. Determine the value of centroid C ($\bar{x}, \bar{y}$).

b. Estimate the coordinate values of B. Was your estimate of the balance point using the pencil close to the coordinate location of the centroid?

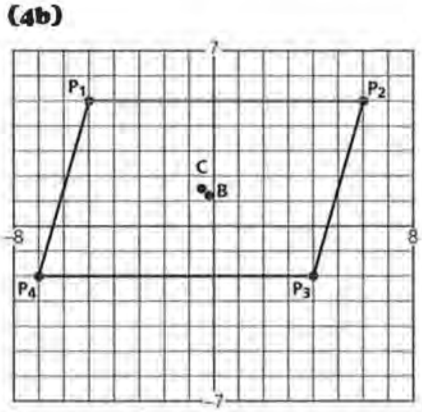
c. Determine the approximate number of centimeters separating your estimate and the calculated location of the centroid C.

d. The centroid of a triangle is located at the intersection of the medians. Do you think the intersection of the medians would locate the centroid for a parallelogram? Why or why not?

A geometry theorem states, "**The diagonals of a parallelogram bisect each other.**" If you have not previously studied this theorem, take some time to work with the parallelogram cut out. In addition to the measurements recorded earlier, draw and measure the segments representing the diagonals. Measure the distances from the intersection of the diagonals to each of the vertices. Keep this theorem in mind as you consider the balance point of four raisins outlining a parallelogram.

Your previous work demonstrated that the midpoint of the segment joining two objects of equal weight represents the balance point. Moving two raisins (or objects) to a position that preserves their balance and combines the weight of the raisins (objects) is called *collapsing the raisins.*

(4b)



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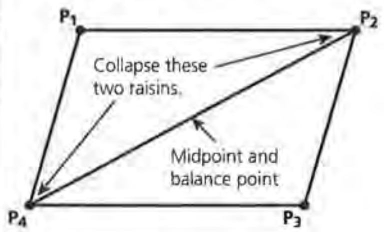
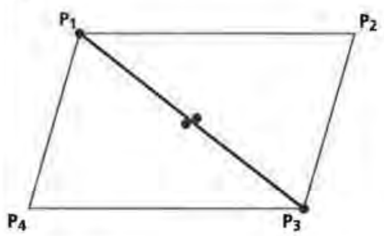
5. Locate the raisins at the midpoint of segment $\overline{P_2P_4}$. The midpoint of the segment connecting two objects of equal weight is the balance point of the two objects.



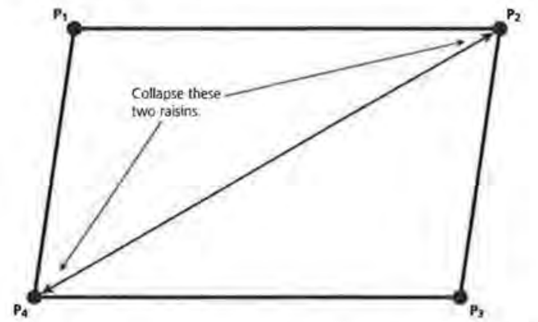
6.



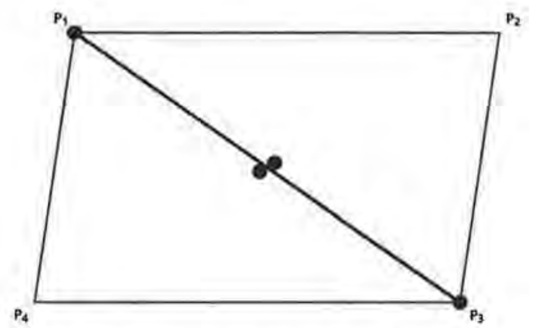
a. The two collapsed raisins will be located at the midpoint of the diagonal $\overline{P_2P_4}$. As the diagonals of a parallelogram bisect each other, this midpoint would also belong to diagonal $\overline{P_1P_3}$.

b. The midpoint of segment $\overline{P_1P_3}$ would balance the new arrangement of the four raisins. Here again, the midpoint of a segment connecting objects of equal weight is the balance point of the objects. The midpoint of $\overline{P_1P_3}$ is the location of the two collapsed raisins and the point that balances the raisins at point P_1 and P_3.

5. Consider raisins located at vertex P_2 and vertex P_4. Also consider the segment connecting these two raisins. Collapse the raisins to produce a new arrangement that continues to balance at B. Describe where the raisins would be located.



6. A sketch of the new arrangement of the four raisins is illustrated below.



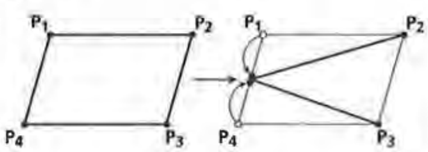
a. Notice the parallelogram is "collapsed" to a segment. Explain why the four raisins would be located along segment $\overline{P_1P_3}$.

b. At what position along this segment would the balance point be located for all four raisins? Why?

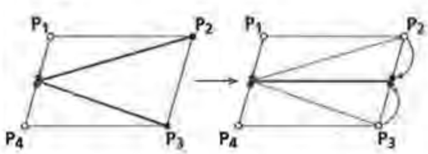
7. The special characteristic that the diagonals bisect each other produces the special point that combines the balance of the two raisins along one diagonal with the balance of the two raisins along the other diagonal. If this characteristic were not true, then the intersection of the diagonals would not locate the balance point. This is further discussed in the Practice section with the isosceles trapezoid as the diagonals of this figure do *not* bisect each other. Emphasize how this special characteristic of a parallelogram is related to the balance point.

8. This problem summarizes what was developed up to this point in the lesson. Students are expected to describe any two of the following methods to determine the balance point: balancing a model using poster paper and raisins, calculating the centroid by placing a cutout model in a coordinate system, or determining the location of the intersection of the diagonals.

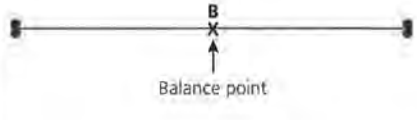
9. a.



b.



10.



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7. How does the balance point of the arrangement illustrated in problem 6 relate to the special characteristics of a parallelogram?

8. If four objects of equal weight outline a parallelogram in a plane, describe two ways you could locate the balance point or centroid.

9. There is another way to determine this balance point. Return to the original arrangement of one raisin located at each vertex P_1, P_2, P_3, and P_4. Develop sketches of the following rearrangements of the four raisins:

a. Collapse the raisins located at P_1 and P_4.

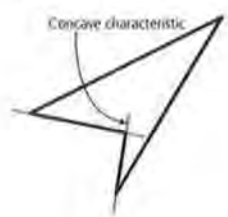
b. Collapse the raisins located at P_2 and P_3.

10. According to your last sketch, two raisins are located at the midpoint of $\overline{P_1P_4}$ and two raisins at the midpoint of $\overline{P_2P_3}$. The midpoint of the segment connecting the two piles of raisins represents the balance point of the figure. Label this point on your sketch.

11. The balance point found in problem 10 represents the same point described in problem 6. What does this indicate about the segment joining the midpoints of the opposite sides of a parallelogram?

Concave Quadrilaterals

Quadrilateral 3 from *Activity Sheet 5* illustrates a different characteristic from the other quadrilaterals. This example is called a ***concave quadrilateral. Concave*** describes any shape with the characteristic that if you extend at least one side of the figure, the extended line would intersect in the interior region of the shape. ***Convex*** describes a shape in which extensions of a side would not intersect in the interior of the shape. Each of the other quadrilaterals included in the options sheet are convex.

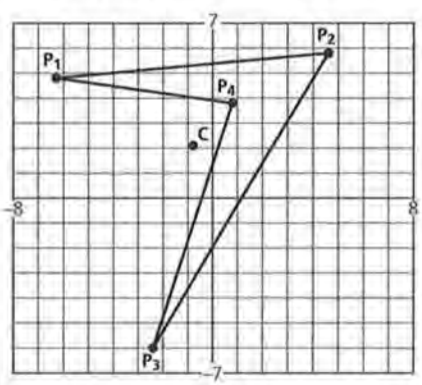


11. The balance point of a parallelogram is located at the midpoint of the segment joining the midpoints of the opposite sides (lots of midpoints here!). This characteristic will also be investigated with other examples of the quadrilaterals. This method works with all quadrilaterals.

Concave Quadrilaterals

12. Students will not be able to balance this quadrilateral with the pencil. As will be investigated further, the balance point is not part of the interior region of the shape.

13. Use a copy of the coordinate system provided with this material and have students place the cutout on the grid. Answers will vary according to the placement of the quadrilateral on a specific grid. One example is provided as a reference for the remaining directions of this problem and problem 14. The location of the centroid is also illustrated.



The following values are approximations of the placement of the boomerang on the coordinate grid:

Point	x-value	y-value
P_1	–6.3	4.8
P_2	4.6	5.8
P_3	–2.4	–6
P_4	0.8	3.8

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12. Cut out Quadrilateral 3 from *Activity Sheet 5* on poster paper. Tape a raisin at each of the vertices P_1, P_2, P_3, and P_4. Using the blunt end of the pencil, attempt to balance the figure. Describe any problems you encounter.

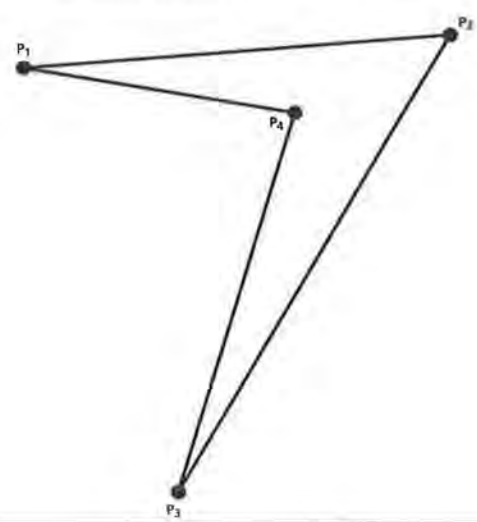
13. Place Quadrilateral 3 on a coordinate grid. Record the coordinate values of each vertex. Copy and complete the following table for this shape.

Point	x-value	y-value
P_1	____	____
P_2	____	____
P_3	____	____
P_4	____	____

14. Using the values recorded on the table, determine the centroid or $C(\bar{x}, \bar{y})$.

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4}{4} \quad \text{and} \quad \bar{y} = \frac{y_1 + y_2 + y_3 + y_4}{4}$$

15. Consider the following steps in locating the balance point of the ***boomerang***. Develop a sketch of the rearrangements of the raisins for each step.



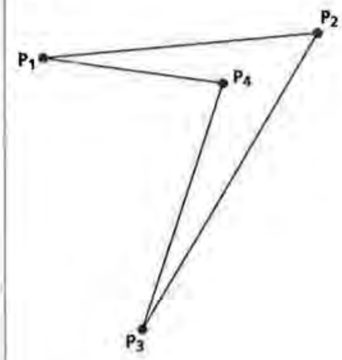
14. $\bar{x} = \dfrac{-6.3 + 4.6 + -2.4 + 0.8}{4} = \dfrac{-3.3}{4}$

$= -0.825$

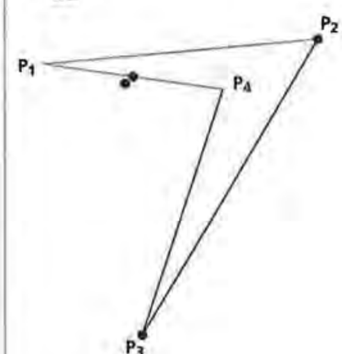
$\bar{y} = \dfrac{4.8 + 5.8 + -6 + 3.8}{4} = \dfrac{8.4}{4} = 2.1$

Centroid is: $C(-0.825, 2.1)$

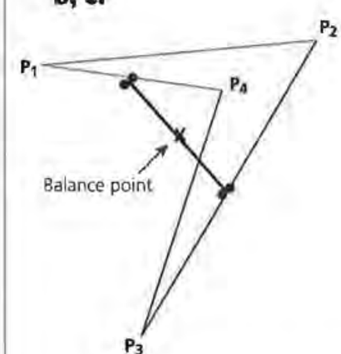
15.



a.



b, c.



c. An estimate of the balance point is indicated in above diagram.

d. The problem with balancing the model on the pencil is that the balance point is located outside the cut-out region of the model.

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a. Collapse the raisins located at P_1 and P_4.

b. Collapse the raisins located at P_2 and P_3.

c. Two raisins are located at the midpoint of P_1P_4 and two raisins at the midpoint of P_2P_3. Sketch an estimate of the balance point.

d. Why did you have a problem balancing the original figure with four raisins?

16. Return the raisins to the points outlining the boomerang. Develop a sketch of the following steps:

a. Collapse the raisins located at P_1 and P_2.

b. Collapse the raisins located at P_3 and P_4.

c. Estimate the balance point.

d. Is this the same location you discovered in problem 15?

e. Summarize how to determine the balance point of a *boomerang.*

SUMMARY

The balance point of four objects (of equal weight) arranged as a quadrilateral can be determined by experimentation and by calculation of the centroid. The balance point of four objects outlining a parallelogram is located at the intersection of the diagonals due to the special characteristics of a parallelogram. Objects of equal weight outlining a quadrilateral have a balance point located at the intersection of the segments connecting the two midpoints of opposite sides. This process works for all quadrilaterals.

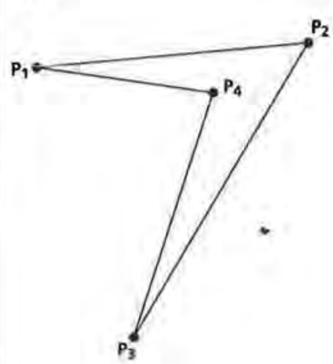
Practice and Applications

17. Consider *Quadrilateral* 2 (the *Isosceles Trapezoid*).

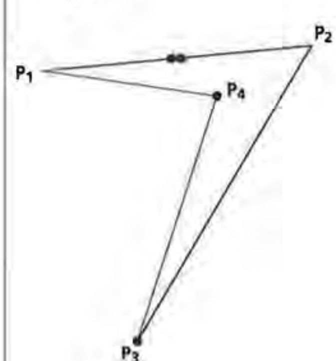
a. Place a cut-out copy of the quadrilateral on a coordinate grid. Copy and record the following information:

Point	*x*-value	*y*-value
P_1	_____	_____
P_2	_____	_____
P_3	_____	_____
P_4	_____	_____

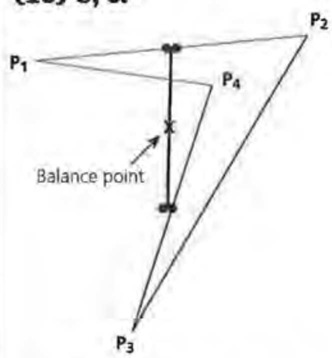
16.



a.



(16) b, c.



c. Estimate of the balance point is the midpoint of the segment connecting the paired raisins. See above diagram.

d. This is the same balance point as determined in problem 15.

e. Problems 15 and 16 determined the balance point by forming the segment connecting the midpoints of the opposite sides. The balance point would be the midpoint of this segment. Emphasize that this process works for both the parallelogram model and the concave model. The location of the balance point outside of the interior region exists in many (but not all) concave models. (It is what makes a concave model "interesting"!)

Note: Combining the results of problems 15 and 16 indicate that the balance point is the intersection of the segments connecting the midpoints of the opposite sides.

Practice and Applications

17. Initially an isosceles trapezoid might appear to have similar characteristics to the parallelogram. This model is introduced to also highlight the special features of the parallelogram. Answers will vary for this problem, however, a specific example of the placement of the isosceles model provides a reference for this problem. (The centroid is also indicated as a reference.)

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b. Why is this figure called an isosceles trapezoid?

c. Describe two methods to determine the balance point of four raisins outlining an isosceles trapezoid.

d. Determine the balance point of this quadrilateral using one of the methods described above.

e. Does the point representing the intersection of the diagonals help you locate the balance point for this quadrilateral? Why or why not?

18. Consider Quadrilateral 4 (the *"Nothing Special" Quadrilateral*).

a. Is this quadrilateral convex? If yes, how do you know?

b. Describe two methods you could develop to determine the balance point of four raisins located at the vertices of this quadrilateral.

19. A theorem involving quadrilateral states:

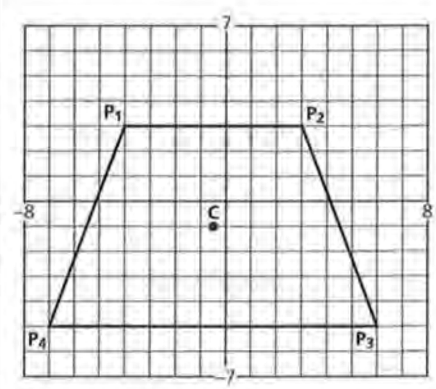
"The figure formed by connecting the midpoints of each side of a quadrilateral is a parallelogram."

a. Sketch an outline of Quadrilateral 4. On this sketch, show the shape formed by attaching a raisin to the midpoint of each segment.

b. Determine the balance point of the four raisins positioned at the midpoints of the sides of Quadrilateral 4. How did you determine this balance point?

c. Is the balance point determined in part b the balance point of the four raisins outlining the pattern of Quadrilateral 4? Explain your answer.

20. Develop a sketch of Quadrilateral 3. Estimate the location of the balance point by the steps outlined in problem 19.



(17) a.

Point	*x*-value	*y*-value
P_1	–4	3
P_2	3	3
P_3	6	–5
P_4	–7	–5

b. First of all, the students might describe the *trapezoid* characteristics, or, a quadrilateral with one and only one set of parallel sides. The *isosceles* characteristic is that the nonparallel sides are congruent.

c. Similar to a student's response in the Investigate section, the balance point could be found by:

- balancing a constructed model (raisins and poster paper) with the pencil;
- calculating the centroid;
- determining the midpoint of the segment connecting the midpoints of opposite sides; or
- determining the intersection of the segments connecting the midpoints of the opposite sides.

d. A calculation of the centroid based on the coordinate values listed in part a is included in the graph. The centroid is C (–0.5,–1).

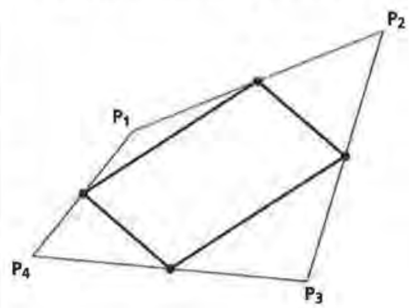
e. A sketch of the diagonals indicates the intersection of the diagonals is not the location of the balance point. The reason is that the diagonals do not bisect each other. (This was a special characteristic of parallelograms.)

18. a. The problem provides an opportunity to emphasize the difference between convex and concave. Quadrilateral 4 is convex as extensions of any of the sides of this quadrilateral do not lie in the interior region of the figure.

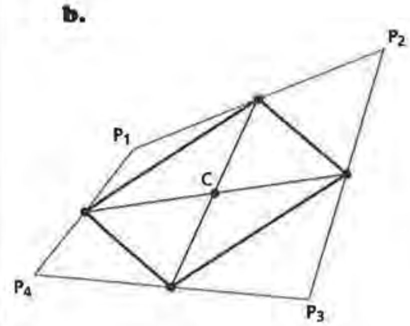
b. The methods previously cited can again be used to determine the balance point by:

- balancing a constructed model (raisins and poster paper) with the pencil;
- calculating the centroid;
- determining the midpoint of the segment connecting the midpoints of opposite sides; or
- determining the intersection of the segments connecting the midpoints of the opposite sides.

19. a. The suggestion of attaching raisins to each midpoint location is done so that students might suggest finding the balance point by balancing the model with the pencil. Recognizing that the raisins form a parallelogram, however, suggests an easier process.



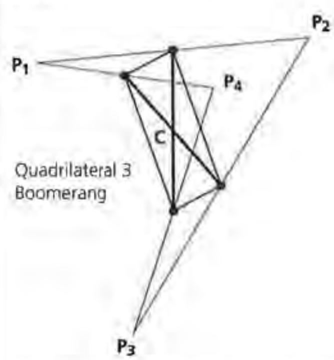
b.



The above estimate of the balance point for the parallelogram formed was based on the intersection of the diagonals of the parallelogram.

c. The balance point estimated in b is the same balance point of Quadrilateral 4. Several explanations could be provided by the students, however, the most obvious (from studying the above figure) is that the balance point of the parallelogram is also the intersection of the segments connecting the midpoints of the opposite sides. This was previously developed as the location of the balance point for any quadrilateral.

20.



Comparing the above location of the balance point to any one of the other methods indicates another process in locating this point.

LESSON 7

Polygons!

Materials: heavy paper, raisins, tape, centimeter ruler, *Activity Sheet 6, Activity Sheet 7*
Technology: graphing calculator
Pacing: 1 to 2 class sessions

Overview

The procedures involved in estimating a balance point by actually balancing a model, calculating a centroid, and "collapsing the raisins" are combined and compared in this lesson. This lesson represents the completion of the process before weighted values are assigned to the vertices of the figure. This lesson particularly highlights the "collapsing" process and compares the location determined by this process to the centers obtained through balance and the centroid. Each procedure emphasizes the balance of distance and weight in the location of a center. This lesson is an excellent follow-up to Lesson 6 and does not require specific work with pentagons or other special polygons.

Teaching Notes

Advanced courses in calculus describe the terms *moments, center of moments, point-mass,* and *lamina.* This lesson attempts to deal with these ideas through experimentation and coordinate geometry. Students are *not* working with these ideas as a calculus topic, however, they are seeing how center for a point-mass model is based on the mean of the coordinate points of each mass comprising the figure. Furthermore, the process of locating the center can be developed by a progressive development of balance points described in the lesson as "collapsing the raisins." Although developed in earlier lessons, this lesson highlights this collapsing process and connects it to the other procedures used in determining a balance center.

Technology

Working with a graphing calculator and the LIST options frequently mentioned in this module are outlined in the student material. It is difficult to predict the latest revisions and developments available in the graphing calculator market, therefore, the steps outlined are to be considered general guidelines in working with a representative graphing calculator. Adjustments should be made and highlighted given the type of calculator available for students.

The use of a spreadsheet application and the graphing components associated with most spreadsheet applications would enrich this lesson. A spreadsheet design could be organized resulting in the graph of the polygon and its balance point. This design will be particularly helpful in the next lesson as the weights (or number of raisins) are changed at each vertex.

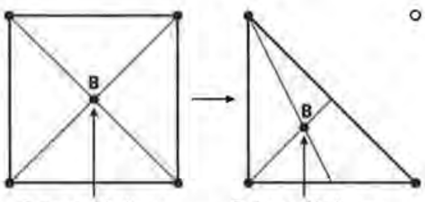
Follow-Up

Students could be directed to a calculus text to locate the topics *moments, point-mass, lamina,* and *center of moments.* They should be reminded they are not studying these topics in the same way as advanced math students involved in calculus would study these topics (which may be very obvious from a review of the particular calculus text obtained by students). Students might, however, be reminded of the importance of these topics as demonstrated by the work in calculus to understand these topics and a review of some of the applications stemming from this study of calculus (i.e., engineering and physics problems).

Solution Key

Discussion and Practice

1.



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LESSON 7

Polygons!

Two flat tabletops are raised off the ground and placed on bricks for support. Would you consider a tabletop with six support bricks more stable than a tabletop with three support bricks?

If one of the support bricks is removed from each tabletop and you are asked to stand on one of the tabletops, which tabletop would you prefer? Why?

OBJECTIVES

Determine the balance point of a point-mass model forming a polygon by "collapsing" the objects.

Connect the method of collapsing the objects to the method of balancing the model using a fulcrum and to the method of calculating the centroid.

Generalize the methods of finding the balance point of the point-mass model.

Completing a study of four objects taped to the poster paper suggests extending an investigation to five or more objects. What happens to the location of a balance point as more objects are taped to the poster paper?

INVESTIGATE

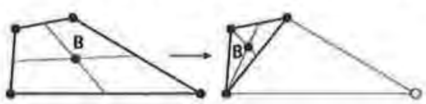
Balancing Multiple Objects

Previous lessons indicated the weight of the raisins, the specific distribution of the raisins, and, to some extent, the weight of the poster paper affected the location of the balance point. How can an estimated balance point be determined for any number of objects taped to a plane?

Discussion and Practice

1. Four raisins were taped to poster paper as indicated in the first of the following two diagrams. One raisin was removed and the shape recut on the poster paper as indicated. Estimate the change in the balance point from removing one raisin by estimating the balance point for each arrangement.

2. Students could be creative and design many arrangements of the raisins. A major change in the balance point would be the result of removing a raisin whose distance from the center of the four raisins is "significant" (use your own judgment on this). The assumption is that the raisins are approximately of equal weight, therefore, the other contributing factor to the balance point is distance. The following is one example of this idea:



3. This problem does not ask (or expect) students to determine the balance point for the pentagon, however, it is an example in which one point has a greater contribution to the balance point due to its distance from the "cluster" of other points. The balance point would be significantly changed by the removal of that point from the model.

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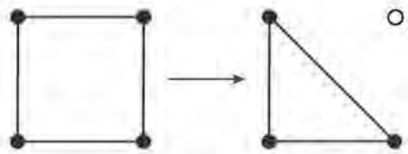
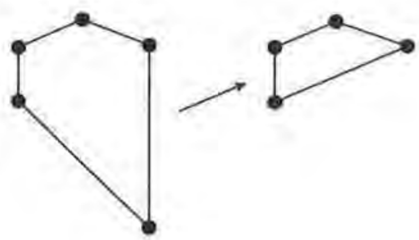


2. Design an arrangement of four raisins on the poster paper that would result in a more noticeable shift in the balance point by removing one raisin and recutting the paper. Indicate in your design which raisin you remove to produce the shift in balance.

The balance point is influenced by the weight of the poster paper, the weight of the raisins, and the distribution of the raisins. The paper represents a uniform distribution of weight (an example of a ***lamina***). Given the type of raisin models investigated in this module, the poster paper has a minor contribution in estimating the balance point. The primary factors in estimating a balance point in the models are the locations and weights of the raisins. As indicated in the previous lessons, this part of the model is called a ***point-mass distribution.***

3. A model of five raisins is illustrated in the following drawing. A raisin is removed and the new shape recut. Indicate if you think the shift in the resulting balance point of the objects would be minor or rather noticeable. Explain your answer.



4. What methods could you use to find the balance point of a point-mass distribution of five or more objects placed on a plane?

4. Students would be expected to cite the method of cutting out the shape, attaching weights (or raisins) to the vertices, and balancing the model with the pencil. Students might also suggest the coordinate geometry approach and the resulting calculation of the centroid.

5. This shape is considered convex as extensions of any of the sides do *not* lie within the interior of the shape.

6. There are very few descriptions for this pentagon other than that it is convex. A measure of the sides indicates two sides are close to the same measure, but this is not a point of classification for pentagons. The term *irregular,* or *not regular,* might be considered a special characteristic, however, this is a pretty unique, convex polygon.

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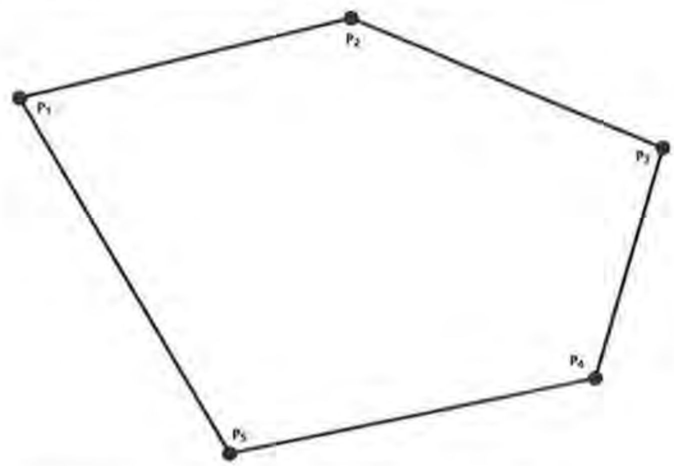
The balance point of five objects of equal weight (raisins) outlining a convex polygon will be investigated in this lesson. A good starting point is the following pentagon (five-sided polygon):



The vertices of this polygon are identified as P_1, P_2, P_3, P_4, and P_5. Trace this figure on another sheet of paper and copy it to poster paper. Make sure you record the labels of the vertices on the poster paper, and then cut out the polygon from the poster paper.

5. Why is this shape considered convex?

6. Are there any special characteristics you previously studied that could describe this pentagon? If yes, explain the characteristics you identified.

You will be investigating several methods to determine the center of balance. Remember the center of balance primarily refers to the center of the distribution of the objects, not the actual cut-out pentagon.

Note: The following comments for problems 7 through 12 are based on the problems directing students to "imagine" the collapsing of the raisins. The location of the points B_1, B_2, B_3, and B_4 are determined through measurements with a centimeter ruler. Actual movement of the raisins and reshaping of the cut-out polygon is not required. If students and teachers are interested in a more hands-on development, consider the Alternate Plan for these problems.

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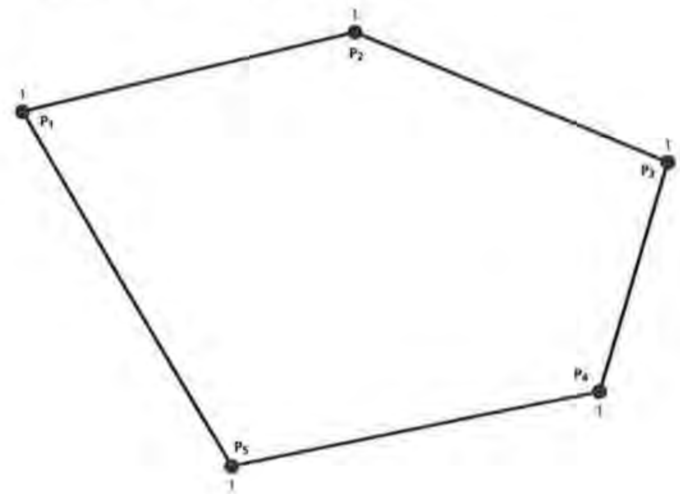
Generalizing a Balance Point

Imagine that five raisins of equal weight are located on the following "map" of the polygon. The raisins will be identified by the vertices of the polygon, or P_1, P_2, P_3, P_4, and P_5. The value "1" on the diagram below indicates the weight of the point-mass (in this case, 1 indicates the weight of 1 raisin).



7. Follow the directions as indicated. Emphasize that the point B_1 is the balance point of the raisins located at P_1 and P_2. By collapsing the two raisins to that one point, the balance of the entire arrangement of five raisins is still maintained.

8. Students follow the directions as indicated in the problem.

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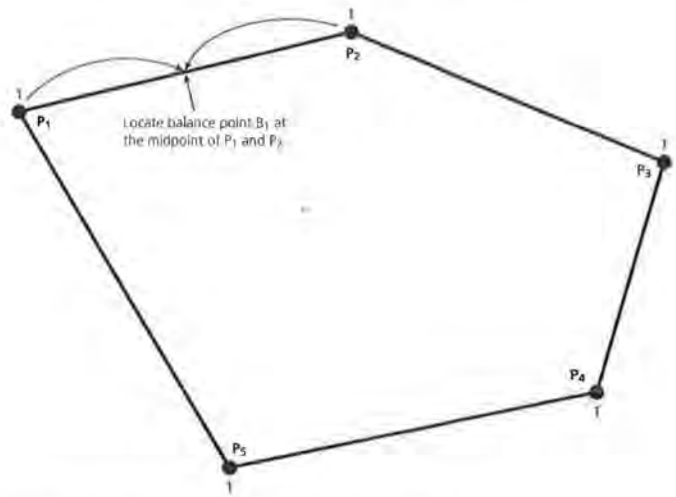
7. Use a ruler to measure the length of $\overline{P_1P_2}$. Previous investigations with two objects of equal weight indicated the balance point for two objects of equal weight is the midpoint of this segment. Determine the midpoint with your ruler and mark this point as B_1 ("balance point 1") on your cut-out model. This step is summarized in the following diagram.



If raisins were actually used in this example, B_1 would represent the location where the two raisins located at P_1 and P_2 can be placed and still keep the original model in balance. Consider removing the raisins from positions P_1 and P_2 and piling them together at B_1. The point B_1 provides a new configuration of the figure. Points P_1 and P_2 have been "collapsed" to point B_1. The pentagon has been "collapsed" into a quadrilateral. However, B_1 represents a weight of two units (or two raisins).

8. With your ruler or straightedge, measure $\overline{B_1P_3}$. Two raisins are located at one end of this segment and one raisin is located at the other end. A balance point involving weights of two units and one unit is needed. (This was previously studied when investigating the balance point of a triangle's medians in Lesson 5.) Let B_2 represent the location of this

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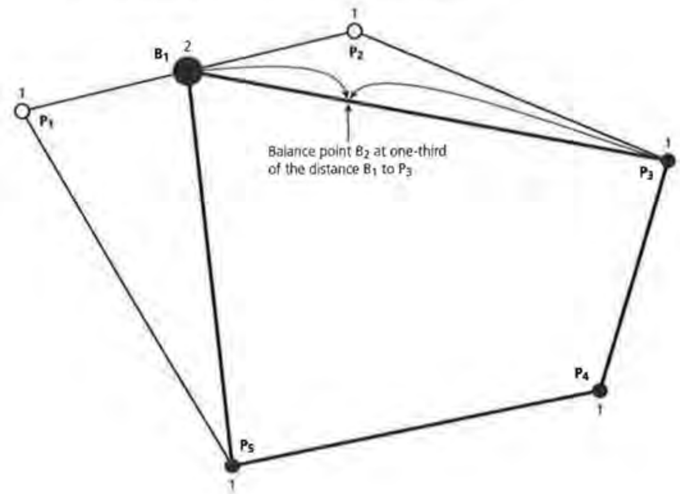
balance point for the two raisins located at B_1 and the one raisin located at P_3. Determine the location of B_2 with your ruler by marking a point one-third of the way from B_1 to P_3. This process is illustrated in the next diagram.



(If necessary, review the steps outlined in Lesson 5 to indicate the location of a balance point one-third of the distance from the object of weight 2.)

Consider removing the two raisins from B_1 and the one raisin from P_3 and piling the three raisins together at location B_2. If B_2 is an accurate location of the balance point for $\overline{B_1P_3}$, the balance point of the original model of the raisins has not changed by collapsing the raisins. The quadrilateral has now collapsed into a triangle with the weight of three raisins at point B_2.

Continue the process. Draw $\overline{B_2P_4}$. The weight at B_2 is three units and the weight at P_4 is one unit. Similar to the work when the raisins were collapsed along the medians of a triangle, the balance point of this segment collapses three raisins at one endpoint and one raisin at the other endpoint.

9. Students follow the directions as indicated in the problem. This problem begins to suggest a pattern. Students found the balance point when three raisins were located along a segment as the point $\frac{1}{3}$ from the endpoint of the two raisins, They are now locating the balance point for four raisins at the location $\frac{1}{4}$ from the endpoint of three raisins.

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Let B_3 represent the point that balances the raisins along $\overline{B_2P_4}$. Also, represent d_1 and d_2 such that:

$d_1 = B_2B_3$
$d_2 = P_4B_3$
and $d_1 + d_2 = B_2P_4$

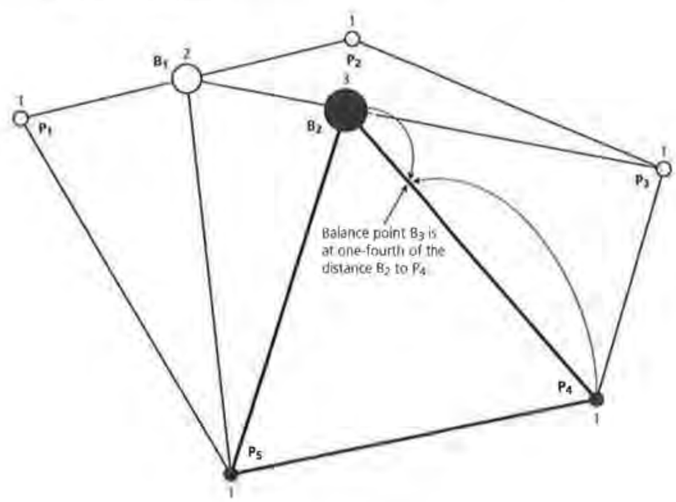
The balance at B_3 indicates that:

$1d_2 = 3d_1$

Therefore,

$d_1 + d_2 = B_2P_4$
$\Rightarrow d_1 + 3d_1 = B_2P_4$ (By substitution)
$\Rightarrow 4d_1 = B_2P_4$
$\Rightarrow d_1 = \frac{1}{4} B_2P_4$

9. Measure segment $\overline{B_2P_4}$. Three raisins are located at one end of the segment and one raisin is located at the other end. Let B_3 represent the location of the balance point for the three raisins located at B_2 and the one raisin located at P_4. Determine B_3 with your ruler by marking a point one-fourth of the way from B_2 to P_4. This process is illustrated in the diagram below.



10. Let B_4 represent the point that balances the raisins along $\overline{B_3P_5}$. Also, represent d_1 and d_2 such that:

$d_1 = B_3B_4$
$d_2 = P_5B_4$
and $d_1 + d_2 = B_3P_5$

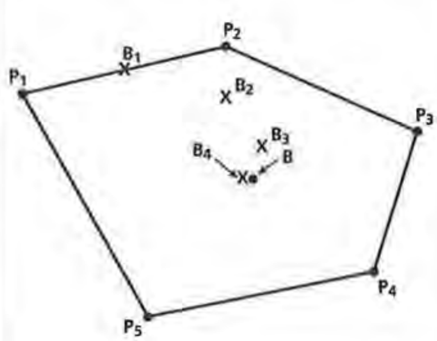
The balance at B_4 indicates that $1d_2 = 4d_1$.

Therefore,
$d_1 + d_2 = B_3P_5$
$d_1 + 4d_1 = B_3P_5$
(By substitution)
$5d_1 = B_3P_5$
$d_1 = \frac{1}{5} B_3P_5$.

11. B_4 represents the balance point of the original arrangement of five raisins. This is further emphasized by comparing locations of B_4 to B (the balance point obtained in problem 12). See the illustration in problem 12.

Balance Point Through Experimentation

12. Students follow the directions as outlined in this problem. The following illustration represents one of the field-tested models. The relative location of B_4 is also labeled in this illustration for use in problem 13.



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10. The process of collapsing raisins is continued. Similar to the previous problems, define d_1 and d_2 to find B_4 (the balance point along $\overline{B_3P_5}$ with four raisins at B_3 and one raisin at P_5).

11. In addition to the balance described in 10, what does B_4 represent?

Balance Point Through Experimentation

12. Tape one raisin to each of the vertices of the original polygon cut out from the poster paper. Using the blunt end (or eraser end) of the pencil used in previous lessons, attempt to actually balance the model on the pencil (or fulcrum). This point represents an approximate balance point of the pentagon model. Label the point on the cut-out figure as point B (the "balance point").

13. Measure and record the distance from B to B_4. Conjecture the meaning of point B and the meaning of point B_4 regarding the original model of the distribution of raisins.

Balance Point Through Coordinate Geometry

14. Place the cut-out polygon on a coordinate grid provided with this lesson. Record the following data based on your placement of the polygon.

Point	x-value	y-value
P_1	____	____
P_2	____	____
P_3	____	____
P_4	____	____
P_5	____	____

15. Determine the centroid C $(\bar{x}, \bar{y})$ of this model by completing the calculations:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} =$$

$$\bar{y} = \frac{y_1 + y_2 + y_3 + y_4 + y_5}{5} =$$

16. Locate the centroid on the coordinate grid. Label this point as C. Also label on the coordinate grid the points B and B_4.

Alternate Plan for 7–12

As indicated, there is a more hands-on approach that some students and teachers might develop for problems 7 through 12. This approach, however, requires a rearranging of the questions. This alternate plan begins by taping one raisin at each of the vertices and balancing the polygon on top of the pencil. The balance point should be labeled as B.

Now begin the collapsing process. First collapse the raisins along $\overline{P_1P_2}$ at point B_1. Direct students to locate the midpoint B_1 and physically move the raisins from P_1 and P_2 to form a "stack" of two raisins at B_1. Reshape the pentagon to form a quadrilateral by cutting out the quadrilateral from the poster paper. This quadrilateral is outlined in the student materials. Balance this quadrilateral with two raisins at B_1 and one raisin at P_3, P_4, and P_5. The resulting balance point should be close to B.

Continue the collapsing process. Direct students to measure one-third of the distance along $\overline{B_1P_3}$ to determine B_2. This again is illustrated in the materials. Physically tape the three raisins involved in the problem to point B_2 forming a "stack" at that point (namely, the stack includes the two raisins from B_1 and the one raisin from P_3). The new arrangement of raisins forms a triangle on the poster paper. Cut out the triangle and balance it on top of the pencil. This balance point again remains close to B. (Variations result from measurement errors, the weights of the raisins, and the poster paper.)

Finally, measure one-fourth of the distance along $\overline{B_2P_4}$ as illustrated in the materials to determine B_3. Rearrange the raisins so that a stack of four raisins is located on top of this point (namely, the three raisins from B_2 and the one raisin from P_4). This stack will require squeezing together the raisins. Cut out of the poster paper a "small strip" representing the segment $\overline{B_3P_5}$. If the poster paper is not completely bent out of shape, attempt to balance the strip with the pencil. This balance point B_4 is again close to B.

This hands-on process was quite effective with some of the field-test groups. It further illustrated the impact resulting from the collapsing process. This plan, however, destroys the original pentagon and requires cutting out another pentagon for completing the other problems.

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SUMMARY

The balance point of objects of equal weight located on a plane can be estimated by designing a model and balancing the model on a fulcrum (or a pencil as used in this lesson). The balance point can also be estimated by placing the model on a coordinate grid and calculating the centroid. Finally, the balance point can be estimated by collapsing the objects (or the raisins) as demonstrated in this lesson.

Practice and Applications

17. The model used in this lesson was a convex polygon.

- **a.** Would determining an estimate to the balance point by collapsing the raisins present a problem with a concave model? Explain.
- **b.** Which method of estimating a balance point might be complicated if the model presented were a concave polygon? Explain.

18. Return to the quadrilaterals studied in Lesson 6. If you have not developed a cutout on poster paper of quadrilateral 4 ("Nothing Special" model), develop one. Using your ruler, determine the position of the balance point by the process described as collapsing the raisins. Label this point as B_3 on the cut-out form.

19. Examine B_3 from question 18 and the balance point obtained in Lesson 6 (or the intersection of the midpoints of opposite sides of the quadrilateral). Label this point as B on the cut-out form if it was not previously labeled. Measure and record the distance between points B and B_3.

20. Determine the balance point of Quadrilateral 3 (the Boomerang model) by the method of collapsing the raisins. Compare the point estimated by this method with the balance point determined in Lesson 6.

13. The distance between B and B_4 in the illustration is approximately 3 mm. B and B_4 both represent the balance point of the pentagon model. It might be explained that B also involves the weight of the poster paper. Again, the uniform distribution of the weight of the paper makes its contribution to the balance point of the model minor.

Balance Point Through Coordinate Geometry

14. Answers will vary depending on the placement of the pentagon on the coordinate grid. One specific example is included with problem 16 and is used as a reference in this problem. Each of the points represents an approximation.

Point	x-value	y-value
P_1	–8	3.8
P_2	0	5.5
P_3	8	2.5
P_4	6	–3.3
P_5	–3	–5.0

15. Determine the centroid C $(\bar{x}, \bar{y})$ of this model by completing the calculations:

$$\bar{x} = \left(\sum_{i=1}^{5} x_i\right)/5 = \frac{-8+0+8+6+-3}{5}$$

$$= \frac{3}{5} = 0.6$$

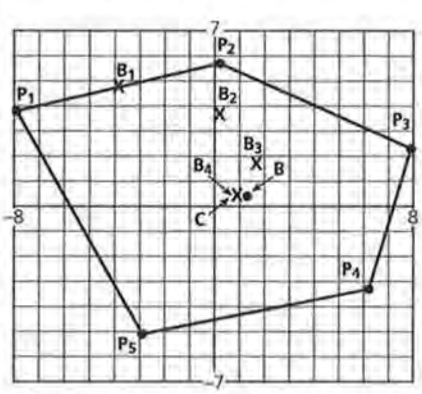
$$\bar{y} = \left(\sum_{i=1}^{5} y_i\right)/5$$

$$= \frac{3.8+5.5+2.5+-3.3+-5}{5}$$

$$= \frac{3.5}{5} = 0.7$$

Location of the centroid is approximately C (0.6, 0.7).

16. The centroid C is very close to the location of B_4. In this example (and in most of the field-tested examples), the locations of B_4, B, and C were very close to each other and demonstrated the different methods to estimate the balance point. In theory, B, C, and B_4 should all be the same.



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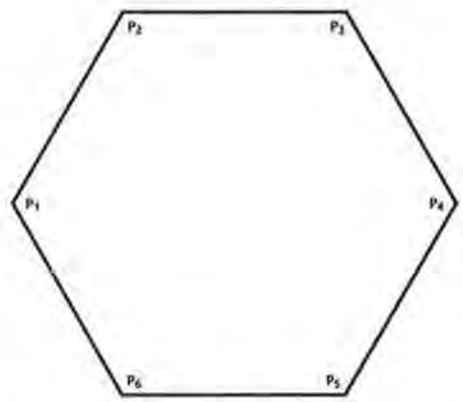
Consider this polygon.



21. What is different about this polygon compared to the model studied in this lesson?

22. Consider a model in which a raisin is taped to each of the vertices of this new polygon. Without using any of the methods presented, sketch (or trace) this polygon and estimate the location of the balance point. (Label your estimate as point E.)

23. What did you consider in making your estimate of the balance point?

24. Determine a balance point for this model by the method outlined in this lesson as "collapsing the raisins." Provide a label for the point you determined by this method. How did this point and your estimate compare?

25. The word *center* is most commonly associated with a circle. How would the center of a circle apply in this example?

Practice and Applications

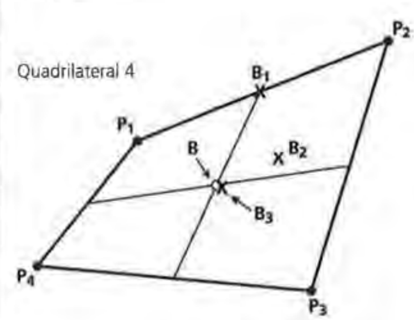
17. The model used in this lesson was a convex polygon.

a. No. Collapsing the raisins should not be influenced by whether or not the model is concave.

b. As the balance point is not always located in the interior region for a concave model, balancing the raisin model on a pencil might not always be possible.

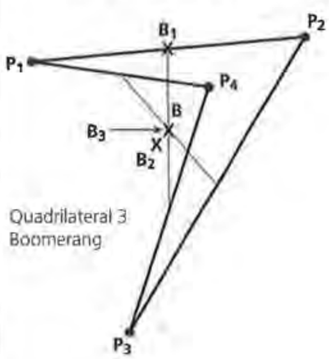
18. Students need to follow directions. This process is illustrated in the following problem.

19.



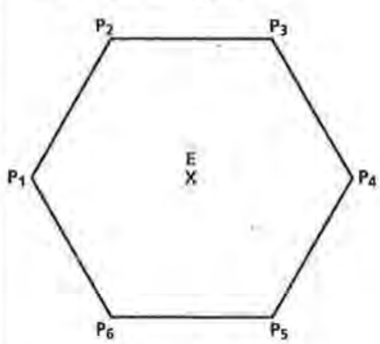
The distance between B and B_3 is very small (hardly measurable!).

20. Again, the estimate of the balance points determined by collapsing the raisins B_3 and by the intersection of the midpoints of opposite sides B are practically the same for the models used in this module.



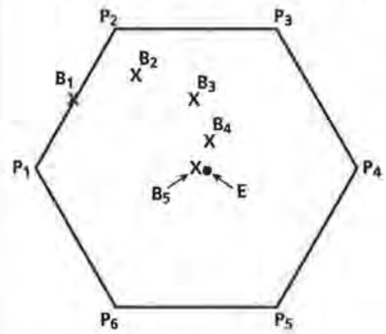
21. This figure is an *approximately regular* shape (one of the first studied by the students).

22. The following estimate of E was based on considering the center of a circle that would have each of the vertices of the polygon lie on the circumference of this circle. (Or, find the midpoint of a segment that would be the diameter of a circle of this type.)



23. The consideration was that a circle could be constructed in which the points would lie on the circumference. This is possible if the figure is a regular polygon.

24. The following example was based on the previous model of the regular hexagon and the estimate of point E. (Each of the points needed in the collapsing process are also provided as a reference for the work involved in this example.)



25. A circle can be constructed around a regular figure. If objects of equal weight were attached to the vertices of this figure, then the balance point, or the "center" of balance, is also the center of the circle constructed around the figure.

LESSON 8

Weighted Means Revisited

Materials: heavy paper, raisins, tape, centimeter ruler, *Activity Sheet 6, Activity Sheet 7,* Unit III Quiz
Technology: graphing calculator
Pacing: 1 to 2 class sessions

Overview

Taping more than one raisin to each of the vertices of the pentagon model will shift the balance point previously determined in Lesson 7. The weighted vertices shift the center in the direction of the heavier pile of raisins. This location can again be estimated by calculating the centroid as directed in the text. The means involved in these calculations, however, require the weighted values be included in the calculations. The specifics of this extension are tied back to the weighted means obtained in Lesson 4. This lesson essentially demonstrates how this is expanded to a two-dimensional model.

Teaching Notes

The raisins taped to the vertices of the pentagon again produce a workable model in estimating the balance point of this figure. The point obtained from balancing the figure on a pencil and the point derived from the weighted means will not be as close as in the previous models. Many factors could be used to explain this (i.e., the varying weights of the raisins or the "spread" of the weights obtained from piling raisins on top of each other at the vertices). Essentially, this lesson completes the models involved in taping raisins as an estimate of locating the balance point. The process was developed as a way to highlight the balance of weight and distance in the problems described to the students.

Technology

Working with a graphing calculator and the LIST options is required in this lesson. The data needed to determine the center are organized in the student material for easy entry into a graphing calculator (specifically the TI-82 or TI-83 models).

As indicated in Lesson 7, the use of a spreadsheet application and the graphing components associated with most spreadsheet applications would enrich this lesson. A spreadsheet design could be organized resulting in the graph of the polygon and its balance point. Varying the raisin count on the spreadsheet and observing the resulting change in the balance point is a powerful way to complete the discussion of this type of center!

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LESSON 8

Weighted Means Revisited

Examine again the pentagon cut out of the poster paper in Lesson 7. How do you think the balance point of the pentagon model would change if the weight placed at vertex P_1 was doubled?

Tape a second raisin to the model at P_1 and balance it with the pencil. Did it change as you predicted?

A more precise location of this balance point is needed to determine the effect of the increased weight at P_1. What makes the calculation of a centroid more difficult if unequal weights are located at the vertices?

OBJECTIVES

Determine the center of balance by calculating the weighted means of the *x*- and *y*-coordinates of objects distributed on a plane.

Interpret the balance point by weights and locations.

Using a pencil to balance a pentagon with a raisin taped at each vertex was a relatively good model to estimate a center of balance of five weights. To what extent does changing the weight of the raisins or their locations on a plane alter the location of the balance point? What if the weight taped at one vertex of the pentagon model was doubled by taping a second raisin over the one already located there? How would this additional weight change the balance point? Similarly, what if a weight were completely removed from a vertex? How would this change the balance point? An earlier lesson taped several raisins at a specific location on a ruler. Increasing the weight at this point altered the location of the fulcrum to balance the new arrangement of raisins. Several questions were investigated related to that balance point. This lesson examines similar questions using a two-dimensional model.

Solution Key

Discussion and Practice

1. A shift in the general direction of P_1 should be noted by an attempt to balance this model. The precise location is not important for this particular problem.

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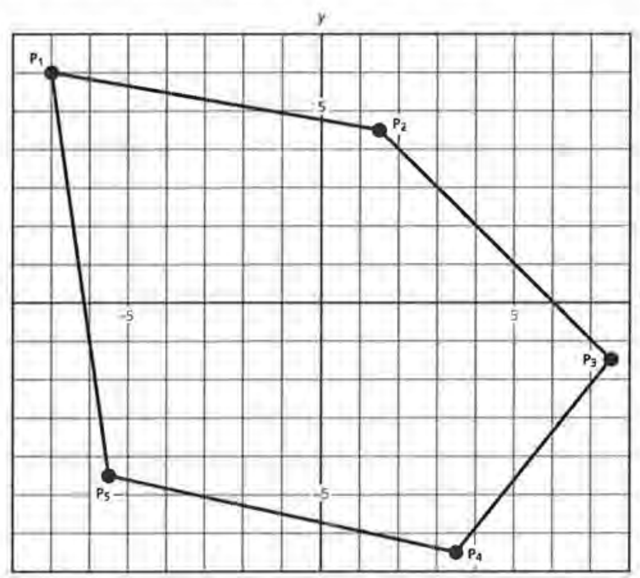
INVESTIGATE

Some Raisins Are Bigger than Others

In this lesson, each vertex will be weighted by *an unequal number of raisins.* Create a total pile of four raisins (one on top of each other) and tape this stack as close as possible to vertex P_1. Similarly, make a total stack of three raisins at P_2, one raisin at P_3, one raisin at P_4, and one raisin at P_5. Although the vertices still trace a pentagon, this new model will have a different center of balance than the model observed in Lesson 7. How would the new center of balance be determined?

Discussion and Practice

1. Estimate the new center of balance by balancing this new model with the pencil. Again, carefully move the pencil on the poster paper until this altered arrangement of raisins balances. How is this center of balance different from the center observed on the previous lesson? Describe the effect the additional raisins produce on the center of balance.



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2. **a.** The horizontal and vertical distances contributed by P_1 are considered four times as great as this point is four times heavier.

b. Similarly, the horizontal and vertical distances contributed by P_2 are considered three times as great as this point is weighted down with a total of three raisins.

Consider the above placement of the pentagon model in an *xy*-coordinate system.

2. Your previous work with a center demonstrated how a balance point equally distributes the ***weighted distances*** along a ruler or number line. A similar balance is suggested by this model except in two dimensions.

a. What horizontal or vertical distance is considered four times as great in this example than in the model presented in Lesson 7? Why?

b. What vertical or horizontal distance is three times greater as a result of the weighted vertices? Why?

Location of the Center of Balance

In addition to balancing the pentagon model on the pencil, the location of the center of balance can be estimated by other methods. The calculation of the centroid as previously explained in this module will provide an estimate of this center. Examine the effect produced by the unequal weights taped to each vertex along the *x*-axis. Based on the placement of the figure as previously indicated, an illustration of the weighted values along this axis is illustrated in the following diagram:

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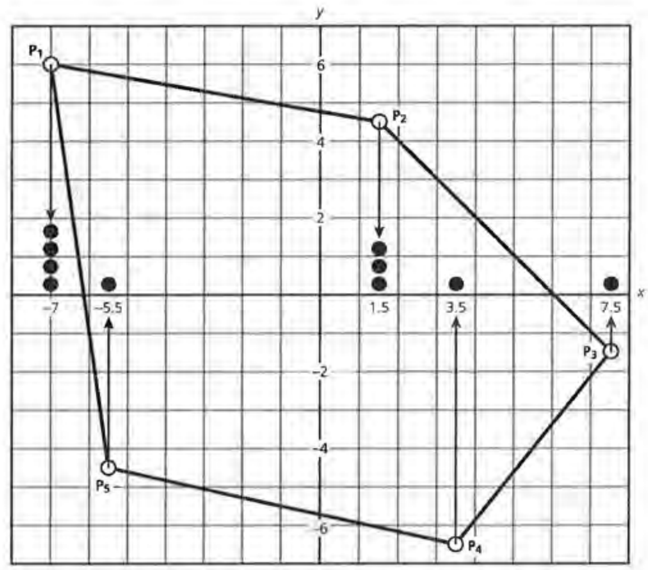

To illustrate the effect of the weights taped at each vertex, the raisins described in this model are "dropped" to their horizontal locations along the x-axis. For example, the four raisins taped to P_1 are stacked on the x-axis at the x-coordinate of P_1. The x-axis now resembles the number line illustrated in Lesson 4. The weighted means of the x-values determines the horizontal position of the new center point. A similar illustration could be developed along the y-axis to represent the vertical position of the center point.

Location of the Center of Balance

3. The chart presented in this problem is intended to guide students into organizing the important data required to find the centroid in two dimensions. This work will be followed up with explanations of using a calculator with LIST options (specifically the TI-83) or a spreadsheet. The formulas will be explained more thoroughly as the calculator is introduced. For this problem, the "formulas" are presented only as an important way to organize the table.

Answers indicating the y-coordinate values may vary from those recorded in the following table. Students should not be too concerned about decimal approximations for those points that clearly do not have integer coordinates. The field-test groups estimated values to the nearest one-half.

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3. Copy and complete the following table based on this particular placement of the pentagon:

Points $P_i (x_i, y_i)$	Number of Raisins W_i $\boxed{L_1}$	x-coordinate values x_i $\boxed{L_2}$	y-coordinate values y_i $\boxed{L_3}$	Weighted x-values $W_i x_i$ $\boxed{L_4 = L_1 * L_2}$	Weighted y-values $W_i y_i$ $\boxed{L_5 = L_1 * L_3}$
P_1 (−7, 6)	4	−7	6	−28	24
P_2 (1.5, y_2)	3	1.5	y_2	4.5	$3y_2$
P_3 (7.5, y_3)	1	7.5	____	____	____
P_4 (3.5, y_4)	1	3.5	____	____	____
P_5 (−5.5, y_5)	1	−5.5	____	____	____

Five points are involved in this example. The weighted mean of the x-values would be represented by the following summation:

$$\bar{x} = \frac{\sum_{i=1}^{5} W_i x_i}{\sum_{i=1}^{5} W_i} = \frac{4(-7) + 3(1.5) + 1(7.5) + 1(3.5) + 1(-5.5)}{(4 + 3 + 1 + 1 + 1)}$$

The x-coordinate of −7 is multiplied by 4 as P_1 is weighted down with four raisins. Similarly, the x-value of 1.5 for P_2 is multiplied by 3 as three raisins are located at this location.

4. The weighted mean for this model is based on dividing the weighted distances by 10. What does the 10 represent?

5. Another way to represent this mean is:

$$\bar{x} = \frac{-7 + -7 + -7 + -7 + 1.5 + 1.5 + 1.5 + 7.5 + 3.5 + -5.5}{10}$$

Why is −7 added four times in this calculation of the mean $\bar{x}$?

6. Complete the calculation of $\bar{x}$.

The weighted mean of the y-coordinate values is used to determine $\bar{y}$ of the center of balance. A summation of the data recorded in the table to calculate $\bar{y}$ is summarized below:

$$\bar{y} = \frac{\sum_{i=1}^{5} W_i y_i}{\sum_{i=1}^{5} W_i} = \frac{4(6) + 3(y_2) + 1(y_3) + 1(y_4) + 1(y_5)}{(4 + 3 + 1 + 1 + 1)}$$

Points $P_i (x_i, y_i)$	Number of raisins W_i $\boxed{L_1}$	x-coordinate values x_i $\boxed{L_2}$	y-coordinate values y_i $\boxed{L_3}$	Weighted x-values $W_i x_i$ $\boxed{L_4 = L_1 * L_2}$	Weighted y-values $W_i y_i$ $\boxed{L_5 = L_1 * L_3}$
P_1 (−7, 6)	4	−7	6	−28	24
P_2(1.5, 4.5)	3	1.5	4.5	4.5	3(4.5) = 13.5
P_3(7.5, −1.5)	1	7.5	−1.5	7.5	−1.5
P_4(3.5, −6.5)	1	3.5	−6.5	3.5	−6.5
P_5(−5.5, −4.5)	1	−5.5	−4.5	−5.5	−4.5

4. The value of 10 represents the total number of raisins, or unit of weights, involved in this model.

5. The value of −7 is multiplied by four as this point is weighted down by four raisins or units of weight.

6. $\bar{x} = \frac{-18}{10} = -1.8$

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7. Complete this calculation of $\bar{y}$.

8. Combine your results from problems 6 and 7. Mark on the cut-out model of the pentagon constructed in Lesson 7 the centroid of this ***weighted mean*** example, or C_w $(\bar{x}, \bar{y})$.

9. Your pentagon model or cutout is rather "crowded" with points. In Lesson 7, estimates of the balance point with one raisin taped to each vertex were identified as B_4, C, and B. Describe how C_w compares to any of these estimates. Did you expect a change in the position of C_w when compared to B_4, C, or B? Explain your answer.

10. How does C_w compare to the location of the balance point obtained with the pencil in problem 1? Was the balance point in problem 1 a good estimate of the new centroid? Explain your answer.

Using a Spreadsheet or Calculator

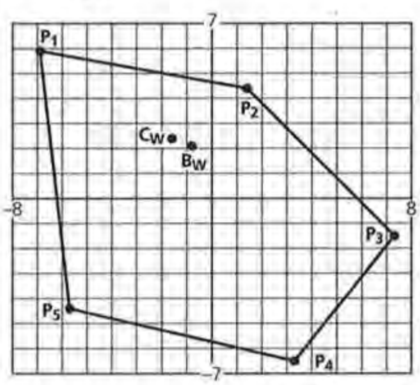
Applications involving weighted means will be more extensively developed in the population models presented in Lessons 9 and 10. Organizing the data involved in these calculations is important in order to determine an accurate centroid. The pentagon example provides an excellent problem to experiment with using either a general spreadsheet (or similar application program) or a calculator. The table presented in problem 3 had columns subtitled L_1, L_2, L_3, L_4, and L_5. Several calculators are equipped with LIST capabilities. The TI-83 (and models developed since the introduction of this calculator) identify the available data lists as L_1 to L_6. Complete the steps as outlined. Although the steps specifically refer to the TI-83 calculator, study the directions as presented. Modifications for other calculators will require understanding the layout of that specific calculator.

11. Enter the number of raisins taped to each vertex in your first list, (or L_1 in the TI-83 setup), the x-values of the vertices in the second list, or L_2, and the y-values of the vertices in the third list, or L_3.

For the TI-83, this is accomplished in the following way: Hit the STAT key. If lists have been previously entered into the calculator, you may need to select the ClrList option of this menu and identify the lists to be cleared. Otherwise, select the EDIT menu option. This allows you to enter your data. A general summary of these steps is developed below. Again, different

7. $\bar{y} = \frac{25}{10} = 2.5$

8. The point C_w is indicated as is the point B_w. B_w was obtained by one of the field-tested models from balancing the pentagon with the pencil. (See problem 10.) The observation from balancing the model should *not* be viewed as a precise location of the balance point. The varied weights of the raisins, the specific location of the taped raisins, and the way in which a student stacks the raisins will contribute more noticeable variations than in the previous hands-on activities. It remains, however, an excellent model for students to experiment with weighted means as it visibly provides them with the anticipated shift in the balance point. Raisins were selected for a number of reasons, including their ability to be squeezed together to provide a stack at the vertices indicated.



9. Students should observe the shift in the balance point to the heavier end of the model, namely a shift in the balance point toward points P_1 and P_2.

10. See above graph. The location of the point B_w will depend on a student's model. The one used in this illustration (rather typical of the field-test group) indicates the calculated shift C_w and the observed physical shift B_w in the balance point.

Using a Spreadsheet or Calculator

11. Students follow directions as indicated in this problem.

12. Again, students follow directions as indicated in this problem. The values observed in the lists should be similar to what students recorded in Problem 3. Developing formulas through this process is very important as it demonstrates the calculator's ability to make many calculations needed in this problem. This becomes very important as students move to the next section and the greater number of data items and calculations.

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calculators will require a modified process in developing lists of the data.

If you need to clear the data from the lists, then . . .	If it is not necessary to clear the lists, then . . .
[STAT] → ClrList [ENTER] ClrList $L_1, L_2, L_3, L_4, L_5, L_6$ [ENTER] [STAT] → EDIT enter data as indicated in the appropriate list columns.	[STAT] → EDIT enter data as indicated in the appropriate list columns.

(Note: to back up, start over, or re-enter the data, you might need to return to the beginning of this process. To start again, hit [2nd] [MODE]. The [2nd] key indicates the command written on top of the raised [MODE] key will be executed or entered. In this case, [2nd] [MODE] indicates the [QUIT] instruction will be executed. This instruction clears the screen and allows you to return to the [STAT] options, or, to re-enter the EDIT option of LISTS.

12. If the data is correctly entered in lists L_1, L_2, and L_3, you are now ready to "program" the weighted values into your fourth and fifth lists, L_4 and L_5. This can be done in different ways. One of the options available for the TI-83 involves the following:

With the lists visible in your window, use the arrow keys, [▶] and [▲], to move the cursor to the top of L_4. If successful at this point in the process, L_4 will be highlighted. The bottom of the screen should display:

$L_4 =$

Using [2nd] 1, and so forth, enter the following formula for L_4:

$L_1 * L_2$ [ENTER]

This formula multiplies the number of raisins and the x-coordinate values for each point. Specific values should now be displayed in the L_4 list.

13. Students follow the directions specified in this problem.

14. Students follow the directions specified in this problem.

The resulting value is $\bar{x}$. This should again agree with the value determined in problem 6.

15. Students should be able to redesign the above process to determine $\bar{y}$. This value should also agree with the value determined in problem 7.

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13. In a similar way, calculate the values for L_5. Use the arrow keys to position the cursor on top of L_5. Enter the following formula for L_5:

$L_1 {*} L_3$ [ENTER]

The values representing the number of raisins multiplied by the *y*-coordinate values for each point should now be displayed in the L_5 list.

14. To find the coordinate values of the centroid, or C $(\bar{x}, \bar{y})$, the following summations are required:

$\bar{x} = \frac{\text{sum of } L_4}{\text{sum of } L_1}$ and $\bar{y} = \frac{\text{sum of } L_5}{\text{sum of } L_1}$

The following steps should be followed to calculate the above values on a TI-83.

- Clear the screen: 2nd [MODE] or [QUIT].
- Hit 2nd [STAT] or [LIST].
- Select MATH from the menu options.
- Select 5:**sum**(from the MATH menu options and then enter L_4).

On the home screen you should see **sum** (L_4). This expression will determine the sum of the values entered in L_4.

- Now hit the divide key and you should see:

  sum (L_4) /

- Hit 2nd [STAT] or [LIST].
- Select MATH from the menu options.
- Select 5:**sum**(from the MATH menu options and then enter L_1).

You should now see **sum (L4) / sum (L1)**.

This indicates the sum of the values entered in L_4 will be divided by the sum of the values entered in L_1.

The resulting value is $\bar{x}$.

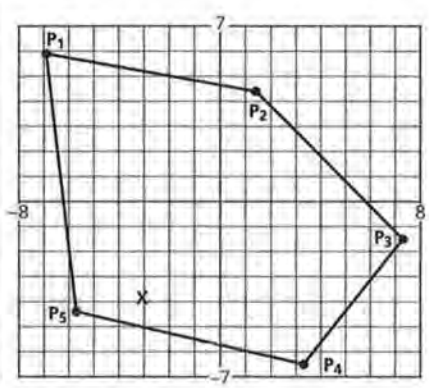
15. Develop similar steps to determine the value of $\bar{y}$ using a calculator or a computer program.

(Note: if problems develop in data entry or sequencing the above steps, hit 2nd [MODE] or [QUIT] to clear the screen and re-enter the steps explained in the previous problem.)

Practice and Applications

16. Answers will vary. This part of the Practice and Applications section encourages students to predict what changes in the weights at each vertex would shift the balance point as indicated. However, if they do not see this process yet, let them speculate on any new arrangement of weights. They will work through the calculation of the resulting balance point (or problem 17 and see the effect of their predictions. The suggested values for each of these cases are one of one many possible responses anticipated from students recognizing the effect of the unequal weights at the vertices.

a.



The following estimates for the number of raisins will be used in problem 17a:

$P_1 = 1$

$P_2 = 1$

$P_3 = 1$

$P_4 = 3$

$P_5 = 4$

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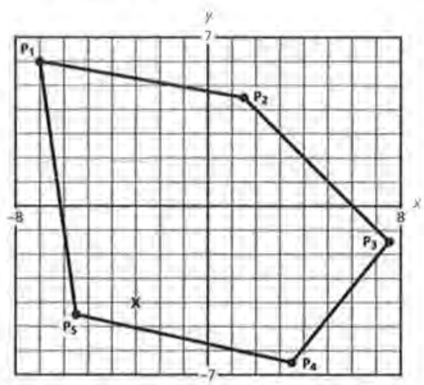
SUMMARY

A balance point for a distribution of weighted objects on a plane is determined by the weighted means of the coordinate values. The difficulty of designing a model to pinpoint the point-mass objects makes the center of balance by experimentation less accurate here than in the previous models demonstrated. The location of the centroid is primarily influenced by the location of the "heavier" points.

Practice and Applications

16. Consider the pentagon investigated in this lesson. *Hypothesize* the number of raisins located at each vertex of this shape that would place the center of balance at approximately the location identified as X for each of the following examples:



a. Estimate the number of raisins at each vertex for this example:

P_1 P_2

P_3 P_4

P_5

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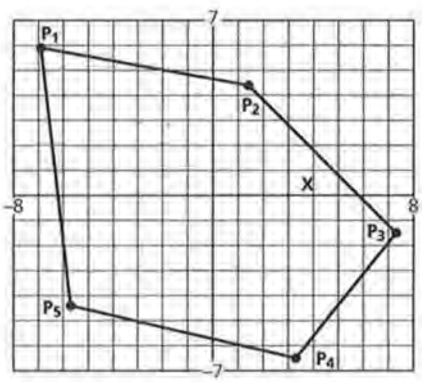
b.



The following estimates for the number of raisins will be used in problem 17b:

$P_1 = 1$

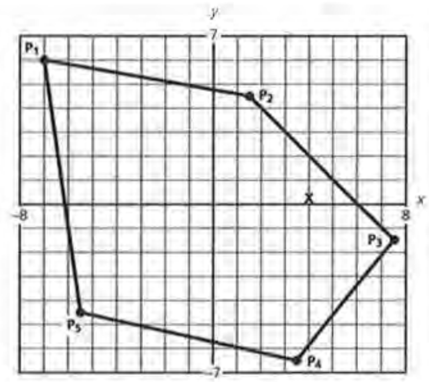
$P_2 = 3$

$P_3 = 4$

$P_4 = 1$

$P_5 = 1$

17. a. The chart is based on the estimates indicated in problem 16a.



b. Estimate the number of raisins at each vertex for this example:

P_1 P_2

P_3 P_4

P_5

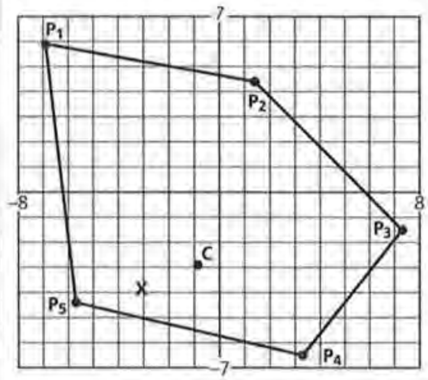
17. Test the accuracy of each of your estimates for problem 16 by calculating the centroid; organize the data for your estimate in a table as outlined below. Copy and complete the following table for each example:

Points $P_i (x_i, y_i)$	Number of Raisins W_i $\boxed{L_1}$	x-coordinate values x_i $\boxed{L_2}$	y-coordinate values y_i $\boxed{L_3}$	Weighted x-values $W_i x_i$ $\boxed{L_4 = L_1 * L_2}$	Weighted y-values $W_i y_i$ $\boxed{L_5 = L_1 * L_3}$
P_1 (–7,6)	____	____	____	____	____
P_2 (1.5, 7)	____	____	____	____	____
P_3 (7.5, 7)	____	____	____	____	____
P_4 (3.5, 7)	____	____	____	____	____
P_5 (–5.5, 7)	____	____	____	____	____

18. Describe how you could improve your estimates for each example.

Points $P_i (x_i, y_i)$	Number of raisins W_i $\boxed{L_1}$	x-coordinate Values x_i $\boxed{L_2}$	y-coordinate Values y_i $\boxed{L_3}$	Weighted x-values $W_i x_i$ $\boxed{L_4 = L_1 * L_2}$	Weighted y-values $W_i y_i$ $\boxed{L_5 = L_1 * L_3}$
P_1 (–7, 6)	1	–7	6	–7	6
P_2(1.5, 4.5)	1	1.5	4.5	1.5	4.5
P_3(7.5, –1,5)	1	7.5	–1.5	7.5	–1.5
P_4(3.5, –6.5)	3	3.5	–6.5	10.5	–19.5
P_5(–5.5, –4.5)	4	–5.5	–4.5	–22	–18

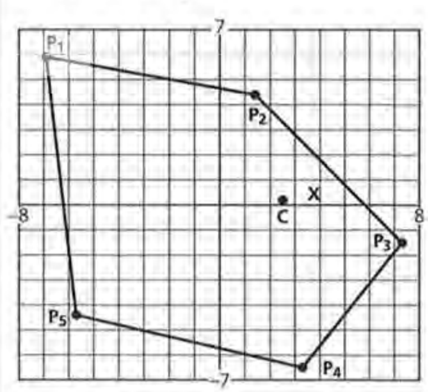
The centroid for the above values is: C (–0.95, –2.85). The graph of this arrangement is illustrated in the following diagram. A greater shift toward P_5 is needed. This could be modified by applying more weights to P_5 or possibly removing a weight from P_4.



b. The following chart is based on the estimates indicated in problem 16b.

Points $P_i (x_i, y_i)$	Number of raisins W_i $\boxed{L_1}$	x-coordinate Values x_i $\boxed{L_2}$	y-coordinate Values y_i $\boxed{L_3}$	Weighted x-values $W_i x_i$ $\boxed{L_4 = L_1 * L_2}$	Weighted y-values $W_i y_i$ $\boxed{L_5 = L_1 * L_3}$
P_1 (–7, 6)	1	–7	6	–7	6
P_2(1.5, 4.5)	3	1.5	4.5	4.5	13.5
P_3(7.5, –1,5)	4	7.5	–1.5	30	–6
P_4(3.5, –6.5)	1	3.5	–6.5	3.5	–6.5
P_5(–5.5, –4.5)	1	–5.5	–4.5	–5.5	–4.5

The centroid for the above values is: C (2.55, 0.25). The graph of this arrangement is illustrated in the following diagram. This arrangement is an excellent guess for producing the balance point indicated. A slight shift toward P_3 might be accomplished by increasing the weight at P_3.



18. See responses to the graphs in problem 17.

UNIT IV

Population Centers

LESSON 9

Finding a Population Center

Materials: No additional materials are required for this lesson.
Technology: graphing calculator
Pacing: 2 to 3 class sessions

Overview

Lesson 9 moves the center determined by a centroid to a *population center.* The models developed with raisins and poster paper (heavier paper) are now representing people and locations on a map. Although the students have studied a center of balance by investigating shapes, the significance of this center is probably still unclear. This lesson involves a map of six communities in southeastern Wisconsin. Each community represented is compared to a vertex in the polygon model of Lesson 8. The population of the communities is similarly compared to the number of raisins taped to each vertex of the polygon model. If a physical model of this map was constructed based on the previous lesson, then a student would create a stack of raisins representing the population of each community on a cut out map of the region. Obviously stacks of raisins representing populations is no longer workable, therefore, the connection of a balance point to a centroid is used to estimate a population center.

The importance of the population center in Lesson 9 is built around the idea of "fairness." By balancing distance and population, a population center represents a location that is fair to the people represented in the region. In a social context, the population center is a way to balance the issues of distance and the number of people.

Teaching Notes

Both this lesson and Lesson 10 are different in style and content from the previous lessons in this module. This is intentional as the focus of the lessons is to bridge the development of the geometric models into specific applications. Although there are several other applications that could have been developed, connecting a center of balance to a population center is a way to connect balance in a social context. (The physical science connections were hopefully implied through the earlier lessons of this module.) A range of questions are developed in the text, however, the field test teachers particularly commented about the interesting discussions and questions related to this topic that were not anticipated.

Technology

This lesson is dependent on use of a calculator or a computer. Without these tools, students will be frustrated. This lesson can be handled by a calculator with LIST options as outlined in the previous lessons. The charts used to support the organization of data needed to complete this lesson follow the model of a TI-83 graphing calculator. Specifically, the six lists needed to calculate the population center are labeled accordingly. Modifying this for other types of calculators or to a computer spreadsheet should not be difficult given the data represented in each list. Exploring several of the "*What if ...?*" questions related to the nature of this lesson would certainly be enhanced by a spreadsheet application or by a calculator with linked list options (for example, a TI-83 or similar product).

Follow-Up

The problems suggested in this lesson are studied in colleges and universities as part of several disciplines. Most notably, several urban centers (similar to Milwaukee, mentioned in the map displayed in this lesson) have universities involved in the study of urban planning. Contact a large university and determine if it has a degree program or a department involved in the estimation of population studies and planning. The opportunity for this type of contact can be investigated by organized searches through the Internet.

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LESSON 9

Finding a Population Center

Suppose there are people, instead of raisins, located at the vertices of a very large pentagon. Would your method change for finding the center of balance?

Why might this center of balance, or population center, be important?

What type of questions does the location of a population center answer?

The balance point models developed in the previous lessons assumed the weight of each raisin (or object) was equal. If several raisins were located at each vertex, then a weighted mean was calculated. This equalizes the effect of each raisin on the balance of the objects.

A *population center* has a similar interpretation. Instead of balancing the weight of objects related to distance, however, a population center balances the "number of people" related to distance. A population center gives "equal status" to each person based on his or her location on a plane. It represents the location where the number of people and their respective distances are balanced. This lesson and Lesson 10 attempt to develop and explain the significance of a population center.

OBJECTIVES

Interpret a population center as a measure of balance.

Set up and calculate a population center.

Interpret a population center as an average of location and population.

Describe the appropriateness of applying a population center to a real-world problem.

INVESTIGATE

Population Centers

A population center is the "balance point" of a distribution of people. Population statistics are extensively studied and analyzed at the local, national, and global level. The United States Constitution directs that an actual count of the citizens of this

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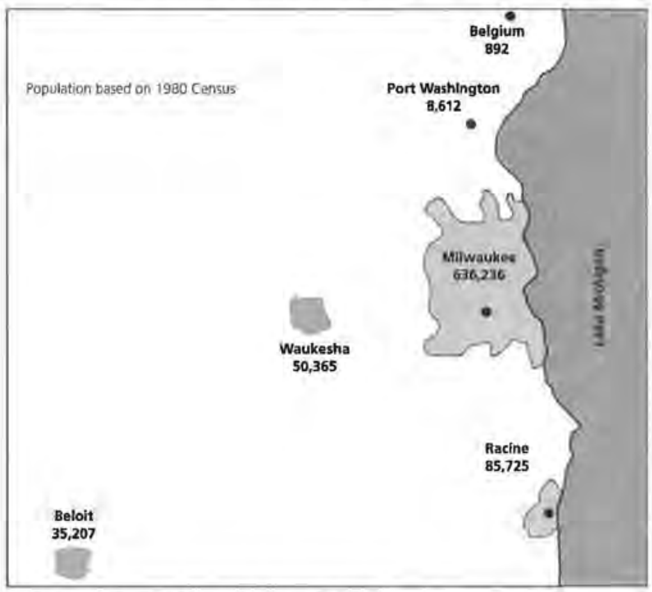
country must be made every 10 years. The Bureau of the Census is the designated agency to carry out a U.S. census at the start of each decade. Each completed census provides volumes of data to interested citizens and political groups.

Discussion and Practice

Pretend you are an important political person back in the 1980s. You are appointed by the Governor of Wisconsin to head a committee to determine the location of an important job service agency in that state. You and your committee are responsible for helping the people residing in the communities of Milwaukee, Waukesha, Port Washington, Belgium, Beloit, and Racine. A map of this area of Wisconsin is provided below. Also included in this sketch are the 1980 population statistics that the committee will use in making a decision. The recommendations of your committee are expected to service the people of this area for at least the next 20 years. A review of your decision will be made in 1990 and again in 2000. You have an



Source: 1980 U.S. Department of Commerce, Bureau of the Census

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interest in running for governor at some future date and feel this appointment is an opportunity to demonstrate your leadership skills.

1. Describe at least three factors you and your committee should consider in making a determination concerning the location of the job service agency.
2. Representatives of the six communities identified as the primary recipients of this service might recommend that the agency be located in their respective communities.
   a. Explain why a decision to locate the agency in Beloit might be "unfair."
   b. Explain why a decision to locate the agency in Belgium might be considered "unfair."
   c. Explain why a decision to locate the agency in Racine might be considered "unfair."
3. If the advisory group attempts to base their decision on fairness to the people located in these communities, what factors do you think are most critical to consider?

Locating a Population Center

The general location of each community is identified on the following map by a point. The collection of the communities (or points) resembles the model of a polygon. Replacing the number of raisins at each vertex with the number of people counted at each community allows you to begin thinking about a balance point. As the population of each community is not equal, calculating a balance point or population center would be similar to the model developed in Lesson 8 for weighted means.

Solution Key

Discussion and Practice

1. Answers will obviously vary at this point. Most students in the field-test group indicated they would locate the agency between Waukesha and Milwaukee. Their explanations included locating the agency closest to most of the people represented in the map, finding a "central" geographical location, and balancing the extreme locations of Belgium and Racine.

2. **a.** Most of the people in the region would be traveling to an extreme southwest location.

   b. More of the people represented in the map would be traveling to an extreme location to the north. Also, Belgium has the smallest number of people; locating the agency in Belgium would require most of the people to travel.

   c. Most of the people in the map would be traveling to an extreme southeast location.

   (Note: An additional challenging question would be to ask if locating the agency in Milwaukee would be fair. Students sense it represents a good location for most of the people, however, they also feel uncomfortable placing the agency in any of the specific cities. It represents a good discussion question for setting up the rest of this lesson.)

3. "Fairness" is a very difficult concept to define; this question could spark an interesting debate. Issues of fairness are very complex and involve many factors not mentioned in problems of this type. Given the data, however, students will probably highlight that fairness seems to be related to distance and the number of people involved in the represented region.

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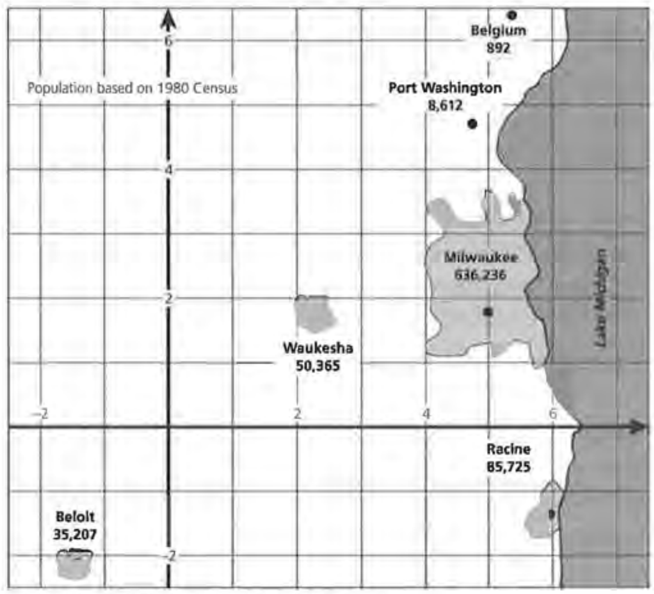
Locating a Population Center

4. The following values approximate the locations.

Consider calculating a population center by placing a coordinate grid over the map below:



Source: 1980 U.S. Department of Commerce, Bureau of the Census

4. Based on this particular coordinate grid, determine approximate x and y values for each of the communities.

Community	*x*-coordinate value	*y*-coordinate value
Belgium	____	____
Port Wash.	____	____
Waukesha	____	____
Milwaukee	____	____
Beloit	____	____
Racine	____	____

Community	*x*-coordinate value	*y*-coordinate value
Belgium	5.3	6.2
Port Washington	4.8	4.5
Waukesha	2.3	1.7
Milwaukee	5.0	1.7
Beloit	–1.5	–2.1
Racine	6.3	–1.5

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5. The values for the last two columns, $P_i x_i$ and $P_i y_i$, have been rounded to the nearest integer values. (See table at the bottom of this page.)

6. a. $\sum_{i=1}^{6} P_i$ represents the sum of the populations of the six cities represented in this region.

$\sum_{i=1}^{5} W_i$ represents the sum of the raisins from each vertex of the pentagon. In effect, we are equating "population" to the "number of raisins."

b. $\sum_{i=1}^{6} P_i x_i$ represents the sum of the products of each population times its location. (Or in this case, it represents the sum of the products of each city's population times its x-coordinate.)

$\sum_{i=1}^{5} W_i x_i$ represents the sum of the products of the number of raisins at each vertex times the corresponding x-coordinate in the pentagon model.

5. A population center organizes the population and coordinate values of the communities in the same way the weights and coordinate values of the raisins were organized in Lesson 8. Copy and complete the following chart to determine the population center. (Note: use of a calculator or spreadsheet is encouraged but not required.)

Community	Population P_i	x-coordinate value (x_i)	y-coordinate value (y_i)	$P_i x_i$	$P_i y_i$
Belgium	892	____	____	____	____
Port Wash.	8,612	____	____	____	____
Waukesha	50,365	____	____	____	____
Milwaukee	636,236	____	____	____	____
Beloit	35,207	____	____	____	____
Racine	85,725	____	____	____	____

6. The population center is described by the following weighted means:

$$\bar{x} = \frac{\sum_{i=1}^{6} P_i x_i}{\sum_{i=1}^{6} P_i} \text{ and } \bar{y} = \frac{\sum_{i=1}^{6} P_i y_i}{\sum_{i=1}^{6} P_i}.$$

a. What does $\sum_{i=1}^{6} P_i$ represent? (Compare this to $\sum_{i=1}^{5} W_i$ in the pentagon model developed in the previous lesson.)

b. What does $\sum_{i=1}^{6} P_i x_i$ represent? (Compare this to $\sum_{i=1}^{5} W_i x_i$ in the pentagon model.)

7. Complete the calculations to determine the population center, or P $(\bar{x}, \bar{y})$, for the map provided.

8. Does the location suggested by problem 7 seem to be a "fair" location for the job service agency? Why or why not?

Will the design or placement of a specific coordinate grid change the location of the population center?

(5)

Community	Population P_i	x-coordinate value (x_i)	y-coordinate value (y_i)	$P_i x_i$	$P_i y_i$
Belgium	892	5.3	6.2	4,728	5,530
Port Washington	8,612	4.8	4.5	41,338	38,754
Waukesha	50,365	2.3	1.7	115,840	85,621
Milwaukee	636,236	5.0	1.7	3,181,180	1,081,601
Beloit	35,207	–1.5	–2.1	–52,811	–73,935
Racine	85,725	6.3	–1.5	540,068	–128,588

7. P $(\bar{x}, \bar{y})$ is approximately P (4.7, 1.2).

8. In most discussions, the students indicated the above location is fair. This location seems to balance the distance and population values for the region represented by this map.

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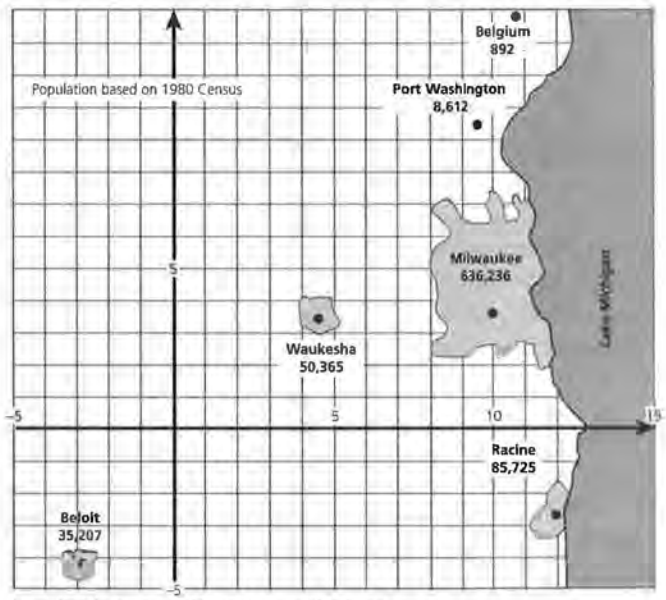
9. The values for the last two columns have been rounded off to the nearest integer. (See table below.)

9. A more detailed coordinate system was designed to determine the population center. A revised map of the communities is provided below.



Source: 1980 U.S. Department of Commerce, Bureau of the Census

Recalculate a population center by collecting and calculating the following information from this revised coordinate grid. Copy and complete this table:

Community	Population P_i	x-coordinate value (x_i)	y-coordinate value (y_i)	P_ix_i	P_iy_i
Belgium	892	____	____	____	____
Port Wash.	8,612	____	____	____	____
Waukesha	50,365	____	____	____	____
Milwaukee	636,236	____	____	____	____
Beloit	35,207	____	____	____	____
Racine	85,725	____	____	____	____

Community	Population P_i	x-coordinate value (x_i)	y-coordinate value (y_i)	P_ix_i	P_iy_i
Belgium	892	10.5	12.3	9,366	10,972
Port Washington	8,612	9.8	9	84,398	77,508
Waukesha	50,365	4.5	3.3	226,643	166,205
Milwaukee	636,236	10.0	3.3	6,362,360	2,099,579
Beloit	35,207	–2.7	–4.0	–95,059	–140,828
Racine	85,725	12.5	–2.8	1,071,563	–240,030

10. Q $(\bar{x}, \bar{y})$ from this table is the point Q (9.4, 2.4).

11. When the locations are plotted on each respective map, the population centers are virtually the same location.

You would expect this as the final averages represent a location based on the locations of the points involved in the problem.

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12. This problem returns to the issue of "fairness." If the primary factors used in evaluating fairness are *location* and the *number of people,* then the population center identified by this process attempts to consider both factors in a fair way.

13. The data for Belgium was omitted from this table. A typical reaction to this is that it is not fair. An "unfair" judgment, however, is based on criteria other than population and location as the remaining problems attempt to point out. Remind students, however, "fairness" is a very complex and an open-ended standard; it is a goal that cannot realistically be defined by this type of problem. This problem is intended to point out the complexity of this concept.

14. As either grid is okay to use in this problem, the following data was collected from the second grid. (Recall values are rounded to the nearest integer in the last two columns.)

10. Determine the population center Q ($\bar{x}$, $\bar{y}$), from this table.

11. Compare the locations of the two population centers derived from each example, P and Q. Would you expect the centers to be similar? Why or why not?

12. If the population center were to be suggested as the location of the agency, do you think each community would consider this recommendation "fair"? In what way does the population center approach seem to address some of the concerns of fairness?

The "Belgium Problem"

An incomplete table of the populations of the communities involved in this decision was developed. Consider the following incomplete table:

Community	Population P_i	x-coordinate value (x_i)	y-coordinate value (y_i)	P_ix_i	P_iy_i
Port Washi	8,612	___	___	___	___
Waukesha	50,365	___	___	___	___
Milwaukee	636,236	___	___	___	___
Beloit	35,207	___	___	___	___
Racine	85,725	___	___	___	___

13. What data were omitted from this table? Is this fair?

14. Determine a population center for the five communities listed in this incomplete table. Use either the first or the second grid to complete the x- and y-coordinate values for each community.

15. Compare the population center derived from this incomplete table with the center derived when all six communities were considered. How does the incomplete data table change the recommendation of the advisory group?

16. Describe what you think is meant by the title of this investigation *"The Belgium Problem."*

17. Consider the omission of the city of Milwaukee data from the table listing the six communities. Would this omission change the location of the center of the population? Estimate where the center of population would be located if the city of Milwaukee were not considered in the population distribution.

Community	Population P_i	x-coordinate value (x_i)	y-coordinate value (y_i)	P_ix_i	P_iy_i
Port Washington	8,612	9.8	9	84,398	77,508
Waukesha	50,365	4.5	3.3	226,643	166,205
Milwaukee	636,236	10.0	3.3	6,362,360	2,099,579
Beloit	35,207	–2.7	–4.0	–95,059	–140,828
Racine	85,725	12.5	–2.8	1,071,563	–240,030

The population center $P(\bar{x}, \bar{y})$ based on the data from the five cities is $P(9.4, 2.4)$.

15. Comparing the population center with the Belgium data to the population center without Belgium does not make any difference. Belgium's very small population (when compared to the other cities in this problem) contributes little to the population center.

16. Essentially the "Belgium Problem" refers to the minor contribution Belgium plays in the location of the population center of this region. This is a good time to summarize exactly what is critical in the calculation of a population center, namely, *location* and *number of people*. If the number of people of a city is minor compared to the other cities, then the only way this city influences a population center is through a location very far from the other cities. As this is not the case, Belgium has little influence in the calculation of the population center.

17. Omitting Milwaukee from the data is far more critical than Belgium. As the number of people is greater than any of the other cities, Milwaukee has the greatest pull in determining the population center of this region. If Milwaukee was dropped from the chart, then the location of the population center for the five remaining cities would be P (7.2, –0.7) using the second coordinate grid illustrated in this lesson. As you study this location on the map, notice how the population center shifts in the direction of the next largest city. Notice also how the locations of the other large cities (i.e., Waukesha and Beloit) have a "pull" on this new population center.

(Note: Another challenging question is to speculate on the role of Belgium in the problem *if* Milwaukee were excluded. Has Belgium increased its contribution to the location of the population center? Have students speculate on that question and how they would test it out. It is anticipated students would examine the population center with and without Belgium to answer that question. Belgium, however, still does not influence the population center.)

Practice and Applications

18. Using the second grid and the 1990 Census of the six communities, the following data would be recorded for this region:

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SUMMARY

The population center represents a location balancing the number of people and their respective locations. Each person is considered "equal" in the calculation of this center. Similar to the previous study of a balance point, the location of the population center is influenced by a region's specific populations and their respective locations.

Practice and Applications

18. Evaluating your 1980 decision is important! The 1990 Census of the six communities studied in this lesson is provided in the following table. Copy the table below. Use one of the two coordinate grids to help you complete this table.

Community	Population P_i	x-coordinate value (x_i)	y-coordinate value (y_i)	P_ix_i	P_iy_i
Belgium	1,405	____	____	____	____
Port Wash.	9,338	____	____	____	____
Waukesha	56,958	____	____	____	____
Milwaukee	628,088	____	____	____	____
Beloit	35,573	____	____	____	____
Racine	84,298	____	____	____	____

19. Before calculating the revised population center using the 1990 population figures, estimate the location of the 1990 population center as a coordinate point on the grid.

Explain what factors you considered in making your estimate.

20. Based on the 1990 population figures, determine the population center P $(\bar{x}, \bar{y})$.

21. Describe how the population center changed from 1980 to 1990.

If the location of the job service agency were based on the 1980 populations, would the six communities be similarly served by the agency in 1990? Explain your answer. Was your 1980 decision a good one?

22. Consider the following investigations. For each item, explain whether or not a population center would be

Community	Population P_i	x-coordinate Value (x_i)	y-coordinate Value (y_i)	P_ix_i	P_iy_i
Belgium	1,405	10.5	12.3	14,753	17,282
Port Washington	9,338	9.8	9.0	91,512	84,042
Waukesha	56,958	4.5	3.3	256,311	187,961
Milwaukee	628,088	10.0	3.3	6,280,880	2,072,690
Beloit	35,573	–2.7	–4.0	–96,047	–142,292
Racine	84,298	12.5	–2.8	1,053,725	–236,034

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considered. In addition, indicate what other factors might be involved in each decision.

a. Location of a church or temple or synagogue for a large congregation.

b. Location of a school for a growing school district.

c. Location of a shopping mall for a suburban community.

d. Location of an airport to service three large cities.

e. Location of a large recreational facility (for example, a stadium for a major league baseball or football team).

f. Location of public housing units for the elderly.

23. Using a map of your state or geographic area, determine the population center for the three or four largest communities of interest to you or your class.

24. Locate Gurnee, Illinois, on a map of Illinois and Wisconsin. This small community is the location of a very large recreational park, Six-Flags Great America®. Why would a very small community be selected as the location for one of the largest amusement parks in the United States?

25. Consider the previous map of the six communities involved in the investigations of this lesson. Someone hypothesized that it was possible for the center of population of the communities to be located in Lake Michigan! Could there be a population distribution of these six communities that would place the center of population in Lake Michigan? Explain your answer.

Before you complete this module, it is very important to more thoroughly understand a population center. For example, a population center is *not* the location that minimizes distances to a specific point for each of the people in the model. Although frequently close to this point, finding a location that minimizes distances represents another type of center. Stay tuned. Investigations in Lessons 11 to 13 will help you understand other types of centers.

19. This will be a difficult problem for the students! The major gains in population are Belgium (which previously did not influence the population center), Port Washington (also a small population city), and Waukesha. Population losses are noted in Milwaukee and Racine. Beloit shows a very small increase. As the location for each city remains the same, the conjecture of a new population center will probably describe a shift to the west or northwest.

20. Based on the 1990 population figures, the population center P $(\overline{x}, \overline{y})$ is P (9.3, 2.4). This is a very slight shift to the west from the previous center.

21. The results indicate little change in the center during the decade. The slight change to the west is probably explained by the increase in the population of Waukesha and the population declines of Milwaukee and Racine. Interestingly, this observation remains a very important factor in planning future service agencies for this area of Wisconsin.

Services for the six communities are well-serviced by the 1980 location as there is little change in the overall population center.

22. a. Probably not a good example of using a regional population center. The congregation of the church, temple, or synagogue would possibly consider a population center of the people belonging to its organization, but this is probably very difficult to determine. Zoning, availability of land, and potential future growth are probably more important factors. Students might research this by contacting a governmental planning agency.

b. Schools need to be located close to the students and to the future growth of a city. This represents a very difficult decision to make. (Many cities have made very poor choices by not considering factors of population growth or decline, locations of residents with children, income levels.) This is primarily a discussion problem that could be answered by considering a population center. In actuality, the issue is clearly more complicated than simply a population center.

c. A population center could be considered in this decision, however, other factors such as availability to freeways or airports, income levels, hotel facilities, future growth of city, availability of land for expansion, and other attractions are very important. Students could research the site selection of the Mall of America in Minneapolis to gain insight into a decision of this type. (A very fascinating and complex set of concerns were considered.)

(22) d. There are airports whose locations were primarily based on a population center of local communities utilizing the airports. Research the airport in Cincinnati to gain an insight into this type of decision. Again, many other issues could be involved in the final decision of where to locate an airport.

e. A decision of where to place a recreational facility is probably based more on access to freeways, airports, environmental concerns, and convenience than to a population center. As a population center can actually be located where there is no population (stay tuned to the next lesson!), a recreational center would probably be located as close to the people attending the facility as possible. This, however, is a very interesting problem for cities working on new stadiums. Students could research a number of cities for problems past or present related to this issue.

f. Housing for the elderly should be located near the elderly—this may or may not be close to the population center of a region. The distribution of elderly citizens would need to be determined.

23. This is an opportunity to explore your own geographical area. This problem could be expanded into locating a regional service center, a shopping mall, an airport, or something similar in which a population center is considered but not necessarily the only factor involved.

24. Gurnee was selected as it is close to the population center of the communities in the southeastern Wisconsin and northeastern Illinois region (essentially the Chicago and Milwaukee areas of Illinois and Wisconsin).

25. It would be possible for the population center to be in or close to Lake Michigan. A very dramatic change in the population distributions would, of course, need to take place. If most of the population of the region were located in Belgium, Port Washington, and Racine, then a hypothetical center in the Lake would be possible! Given the previous work with geometric figures, examine the midpoint of the segment connecting Belgium and Racine. As it lies in Lake Michigan, a population distribution that placed most of the people in Belgium and Racine would have a population center near this midpoint.

The following is an example of a spreadsheet used to calculate most of the values requested in the charts. Notice also the formulas developed to investigate changes in populations:

Community	Population P_i	x-coordinate value	y-coordinate value	$P_i x_i$	$P_i y_i$
Belgium	1,405	10.5	12.3	14,753	17,282
Port Washington	9,338	9.8	9	91,512	84,042
Waukesha	56,958	4.5	3.3	256,311	187,961
Milwaukee	628,088	10	3.3	6,280,880	2,072,690
Beloit	35,573	−2.7	−4	−96,047	−142,292
Racine	84,298	12.5	−2.8	1,053,725	−236,034
				9.32	2.43

Community	Population P_i	x-coordinate value	y-coordinate value	$P_i x_i$	$P_i y_i$
Belgium	1,405	10.5	12.3	=ROUND(B2*C2,0)	=ROUND(B2*D2,0)
Port Washington	9,338	9.8	9	=ROUND(B3*C3,0)	=ROUND(B3*D3,0)
Waukesha	56,958	4.5	3.3	=ROUND(B4*C4,0)	=ROUND(B4*D4,0)
Milwaukee	628,088	10	3.3	=ROUND(B5*C5,0)	=ROUND(B5*D5,0)
Beloit	35,573	−2.7	−4	=ROUND(B6*C6,0)	=ROUND(B6*D6,0)
Racine	84,298	12.5	−2.8	=ROUND(B7*C7,0)	=ROUND(B7*D7,0)
				=SUM(E2:E7)/SUM(B2:B7)	=SUM(F2:F7)/SUM(B2:B7)

LESSON 10

Finding the Population Center of the United States

Materials: *Activity Sheets 8–12,* Unit IV Quiz
Technology: graphing calculator
Pacing: The number of class sessions will vary depending upon the format selected for completing this project.

Overview

This lesson opens up the study of a population center to the larger goal of estimating the population center of the United States using the 1990 Census data. The importance of this location as studied by previous locations of the United States' mean population center are sited in the text. This lesson involves a similar process to the work presented in Lesson 9, however, it expands significantly the organization and impact of the process. This is an excellent miniproject to conclude the group of lessons that started with Lesson 3.

Teaching Notes

There are several ways to complete the project discussed in the text. Essentially, the following options should be considered.

Option 1

One of the teachers in the field-test group organized small groups of three to four students to complete the "*The Big Picture.*" Students were provided atlases to determine the location of state capitals. Each group developed their own coordinate system on the unmarked U.S. map (one of two maps provided with the masters), located the 48 state capitals, and completed the data sheet organized for the "*The Big Picture.*" (See *Activity Sheets 8 and 12.*) By developing this lesson through small groups, this project took three days to complete. (A considerable amount of time was related to the geography connections of locating capitals and learning information about the states!) This teacher was very satisfied with the results from the groups.

Option 2

A coordinate map of the 48 connected states is provided as *Activity Sheet 9.* In addition, approximate locations of each state capital are provided. This map can be presented to students. Estimating the population center involves recording the *x*- and *y*-coordinates for each state's capital and completing the population charts as outlined for this project. This option saves considerable time in locating the capitals. It also provides a check on the accuracy of students' estimates of the *xy*-coordinates. This work could also be developed through small groups.

Option 3

A partially completed data recording sheet is provided as *Activity Sheet 10.* This sheet includes the coordinate values for each state's capital as illustrated by the coordinate map described in Option 2. Students are still expected to complete the chart as outlined in the text. Although some tedious calculations still remain, use of the activity sheets will decrease the time needed to complete this version of the project. (Essentially the connections to geography topics are less developed with this option.)

The significance of the population center is important regardless of which option is selected. Compare the students' estimates of the population center to the results reported by the Bureau of the Census. (Again, *Activity Sheet 12* is provided to indicate the 1990 population center and some background information on this location. Interesting information about Crawford County, Missouri, can also be located on the Internet!). This is an interesting lesson to develop and an interesting lesson for discussion. Developing this project with a U.S. history or geography class was viewed by one of the field-test teachers as an especially exciting extension for both classes.

Technology

If the development of a center involves all of the states, then a spreadsheet application may be required. (The memory of a graphing calculator might be too small to handle this volume of data. Check the specifications of your calculators!)

Follow-Up

A historical study of the locations of several state capitals can be related to estimating a city best representing a state's mean population center. A research of the capitals (through the Internet and through general geography and history reference books) would probably indicate the significance of this type of center in history.

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Solution Key

Discussion and Practice

1. This shift to the west can be explained by the expansion of the country in that direction. A lot of history can be seen in the shifts of the population centers. Land purchases and westward migration are the most obvious explanations for this shift.

LESSON 10

Finding the Population Center of the United States

Do you think you could compute the population center of the entire United States?

Would you expect the population center of the U.S. in the year 1775 to be the same as the population center of the U.S. in the year 1850? Why or why not?

OBJECTIVES

Calculate a population center based on coordinate points and census populations.

Hypothesize population changes and the effects on the population center.

The population center for the six communities in southeastern Wisconsin was a manageable extension of the polygon models. Expand on this idea even more by imagining that you are in outer space looking down at the entire United States. Instead of viewing a small section of one state, you are now looking at an entire country! Where do you think the population center for the United States is currently located?

INVESTIGATE

Westward, Ho!

The U.S. Bureau of the Census determines the population center of the United States from the census data collected at the beginning of each decade. The map on the next page plots the population centers for each decade from 1790 to 1980, as determined by the Bureau of the Census. Consider how the population center of the United States has shifted over the years.

Discussion and Practice

1. The population centers have consistently moved westward. Explain this westward shift in the population center.

2. The migration to the south is due to many factors, including climate, aging of the population (the migration of older people to warmer climates), labor conditions, and the influx of citizens from Mexico into the southwestern regions of the country.

3. Most predictions are based on continued shifts to the west and south for the same reasons mentioned in problem 2.

4. **a.** Westward movement due to the expanding access to this area of the country. (The Oregon Trail and other access routes were opening up this area of the country.)

b. The expansion and inclusion of the western states. (Actual states were admitted to the country during this time.)

c. Labor and climate issues that resulted in numbers of people moving to the west and south at the expense of the "rust" belt regions.

d. The shift noted here is partially due to the inclusion of Alaska and Hawaii into the calculation of the population center.

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2. In addition to a westward shift, the population centers have also indicated a trend to the south, especially since 1910. What are your ideas on the gradual shift to the south and west?

3. Your goal in this lesson is to approximate the center of population for 1990. The location of this center as determined by the 1990 Census will be shared with you at the conclusion of this lesson. Based on the centers presented in the above map, where do you think the 1990 population center was located? Why did you select this location?

Source: *The Statistical Abstract of the United States*, 1991

4. Some decades indicate a greater shift of the country's population center than other decades. Investigate what historical events highlight the following decades that might explain the greater shifts in the population noted on the map:

a. 1870–1880

b. 1900–1910

c. 1970–1980

d. 1950–1960

5. As previously explained in Lesson 9, the population center is a measure that "equalizes" people based on their relative locations on the map of the United States. The population counts that were used in determining the population center were based on the results from the Bureau of the Census for

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each decade. Based on your understanding of the census, consider the following questions.

a. Do you think the Bureau of the Census is currently accurate in counting almost everyone in the country? Why or why not?

b. Do you think the Bureau of the Census was accurate in counting most of the people in the early decades of this map? Why or why not?

c. Describe people the Bureau of the Census might not have counted in the decades of 1790 to 1850.

6. Washington DC was a "planned city." The location of Washington DC was based on an honest attempt to identify a fair location for all the people in the country. Do you think the location of Washington DC was based on the population center of the country? Why or why not? (Research this question!)

How Is the Population Center Determined?

The population center of a country is found by calculating a point that "balances" the country's population. The magnitude of the numbers of people and the distances involved prevent you from developing an exact location of this center, however, the process presented in Lesson 9 could be used to determine an estimate of the population center. The process involved in estimating the 1990 population center will require a map of the 48 states, a list of the 1990 populations for each state, and an atlas or similar resource to locate state capitals.

Before a process is outlined to develop this project, a few questions need to be discussed specifically concerning Alaska and Hawaii.

7. Consider the populations of Alaska and Hawaii.

a. Why are Alaska and Hawaii interesting variations to the rest of the country?

b. Are there any other locations in the United States that might similarly complicate a representation of the population center for all of the citizens?

c. If yes, identify the locations and why they pose a problem.

5. **a.** The Census Bureau has particular difficulty in counting all the people in large urban areas. There is some serious discussion that figures from the Census need to be adjusted. This was determined from sampling studies that investigated the low number of people counted in these areas. It is particularly interesting to point out to students that the Census numbers are very important in determining representation in Congress and the distribution of federal money.

b. Early census counts did not take into account slaves, Native Americans, and probably poor populations. It is impossible to determine exactly how many people were involved in this "under count."

c. As indicated in b, slaves, Native Americans, and the poor were not represented at that time. Questions exist as to whether or not women were adequately counted during some of this time.

6. Thomas Jefferson was the first president inaugurated in Washington DC in 1801. At that time, Washington was close to the estimated population center of the country (see the map from the Census Bureau). Shortly after Jefferson became president, however, our country purchased the Louisiana Territory. This purchase dramatically changed the population center and the future distribution of the population of our country for years to come.

7. **a.** Alaska and Hawaii are not connected to the rest of the states of the country. The 48 states are frequently called the 48 connected states. The distance of Hawaii from the mainland is particularly impressive—thousands of miles!

b.–c. There are small islands and other locations scattered around the world that are considered part of the United States. They are not part, however, of any state. (They are considered territories or trust areas.) Washington DC is also an interesting concern—it is not a state and is not factored into the process outlined in this lesson. Students could easily add it to the process.

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8. Lesson 9 discussed the "Belgium problem." Essentially this problem indicated that the magnitude of the population of the larger communities was so much greater than Belgium's population that Belgium had little impact on the population center. The population sheet listing the 1990 population for each state indicates Alaska and Hawaii are small population states. However, they represent a different concern than simply a small population. In what way do Alaska and Hawaii require a different consideration than other states with small populations?

9. In order for Alaska and Hawaii to be included in the activities outlined in this lesson, an accurate map including Alaska and Hawaii is needed. This type of map is very difficult to develop.

- **a.** Examine a globe and locate Alaska. If you were given the population center of the 48 connected states, how would the addition of Alaska affect that center?
- **b.** Examine your globe again and locate Hawaii. If you were given the population center of the 48 connected states, how would the addition of Hawaii affect that center?
- **c.** The activities outlined in this lesson will conclude with an estimate of the population center for the 48 connected states. How would you suggest that center be adjusted to account for inclusion of Alaska and Hawaii?

The model in Lesson 9 formed a collection of points corresponding to the locations of the communities on the map. In addition, each point was weighted according to the population of the community. What will the points represent in the U.S. model? What locations will be used? What population statistics will be used?

The Big Picture

There are several methods that could be used to estimate the center for the 48 connected states. The first activity outlined is called "The Big Picture" method. It involves a lot of work and detailed investigations! This method is most manageable if developed with a spreadsheet. The following steps outline this method:

Step 1: Study a map of the United States and construct a coordinate system on the map. You may use *Activity Sheet 8*

8. The Belgium Problem points out how a small community contributed little to the location of a population center. However, Belgium was small and located close to the other communities. Alaska and Hawaii are small but located far from the rest of the population points. The distances involved contribute to the population center.

9. **a.** Two problems are presented by Alaska. First of all, it is a very large state by area. Estimating a population center for Alaska is its own problem. If Alaska's state capital is used as an estimate of its center, then the large distances from the 48 connected states would suggest that a slight shift of the population center to the northwest would be needed to include Alaska.

b. Hawaii is a tremendous distance to the west (and south) of the 48 connected states. A slight shift to the west and south of the population center for the 48 states might be considered valid to include Hawaii. Again, however, the shift is very slight as the population value is small.

c. Combining the two recommendations would suggest a very slight shift to the west. As the population values are small for the two states, this shift is minor. (Note: The intent is not to make this problem unruly, however, the problem presented by Alaska and Hawaii should be at least discussed. Actual work with a map that included the two states provided little insight in the problem as the coordinate values for the other 48 states were difficult to determine accurately.) Emphasize that the goal is to develop an estimate of the population center. The data and tools for a very accurate location of the population center are more complex than presented in this lesson.

The coordinate values for each state were determined by approximating the general location of the state capitals based on the coordinate system superimposed on the U.S. map. Although an important part of this activity involves finding the locations on the map and recording the coordinate values, it might be possible to provide this data sheet with the accompanying map if the timetable for this lesson needs to be cut.

See table on the next two pages. The partially completed table is also shown on *Activity Sheet 11*.

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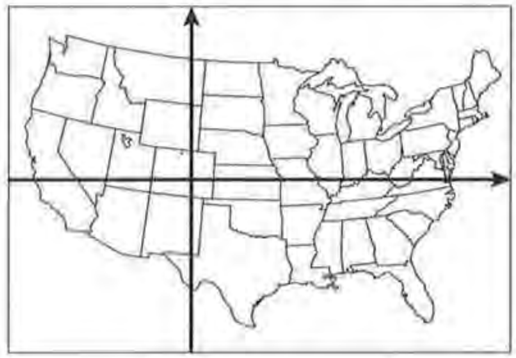
to help you. The specific placement of the origin and the x- and y-axes of this system is not important.



Step 2: Design *accurate* units on both your x- and y-axes. These units will be used to estimate the coordinates of the population centers for each state.

Step 3: Estimate a population center for each state by identifying the location of the state's capital. Many states modeled the federal government in planning a city as their capital based on an estimate of the population center of the state at the time statehood was granted. Clearly some state capitals are not good estimates of a population center, however, the effect on the coordinate values will not significantly change the estimation of the population center.

Step 4: Use the 1990 State Population Chart on *Activity Sheet 10* to help you complete the calculations for the weighted values of each state.

Although not all locations or people are represented by this process, the number of points located on a map indicate a major representation of the country's population. Complete the data requested in the chart on *Activity Sheet 10*, by using the coordinate values obtained from your map. (A spreadsheet is ideal for this method, although some graphing calculators such as the TI-83 are able to handle this large data collection.)

Data for Determining the U.S. Center of Population
"The Big Picture"

(A spreadsheet is ideal for this method. Not all calculators will be able to handle a data set this large. However, this data collection can be handled by a TI-83.)

State	Population P_i (in thousands)	x-coordinate values	y-coordinate values	P_ix_i	P_iy_i
AL	4,041	2	–1.5	8,082	–6,061.5
AR	2,351	–0.5	–0.5	–1,175.5	–1,175.5
AZ	3,665	–8.5	–0.5	–31,152.5	–1,832.5
CA	29,760	–11.5	2.5	–342,240	74,400
CO	3,294	–5.5	2.5	–18,117	8,235
CT	3,287	6	4	19,722	13,148
DE	666	5.5	2.5	3,663	1,665
FL	12,938	2.5	–2	32,345	–25,876
GA	6,478	2.5	–0.5	16,195	–3,239
IA	2,777	–1.5	3	–4,165.5	8,331
ID	1,007	–9	5	–9,063	5,035
IL	11,431	0.5	2	5,715.5	22,862
IN	5,544	1.5	2.5	8,316	13,860
KS	2,478	–2.5	2	–6,195	4,956
KY	3,685	2	1.75	7,370	6,448.75
LA	4,220	0	–2.5	0	–10,550
MA	6,016	6.5	4.5	39,104	27,072
MD	4,781	5.0	2.5	23,905	11,952.5
ME	1,228	7	5.5	8,596	6,754
MI	9,295	2	3.5	18,590	32,532.5
MN	4,375	–1.5	5	–6,562.5	21,875
MO	5,117	–1	1.5	–5,117	7,675.5
MS	2,573	0.5	–1.5	1,286.5	–3,859.5
MT	799	–7.5	6	–5,992.5	4,794

Data for Determining the U.S. Center of Population "The Big Picture" (continued)

State	Population P_i (in thousands)	x-coordinate values	y-coordinate values	P_ix_i	P_iy_i
NC	6,629	4.5	1	29,830.5	6,629
ND	639	−3.5	5.5	−2,236.5	3,514.5
NE	1,578	−2.5	2.5	−3,945	3,945
NH	1,109	6.5	5	7,208.5	5,545
NJ	7,730	5.5	3.25	42,515	25,122.5
NM	1,515	−6	0.5	−9,090	757.5
NV	1,202	−10.5	3	−12,621	3,606
NY	17,990	5.5	4.5	98,945	80,955
OH	10,847	2.5	2.5	27,117.5	27,117.5
OK	3,146	−3	0	−9,438	0
OR	2,842	−11.3	6	−32,114.6	17,052
PA	11,882	5	3	59,410	35,646
RI	1,003	6.5	4.25	6,519.5	4,262.75
SC	3,487	4	0	13,948	0
SD	696	−3.5	4.5	−2,436	3,132
TN	4,877	1.5	0.5	7,315.5	2,438.5
TX	16,987	−3.0	−2.5	−50,961	−42,467.5
UT	1,723	−7.75	3	−13,353.25	5,169
VA	6,187	5	1.5	30,935	9,280.5
VT	563	6	5.5	3,378	3,096.5
WA	4,867	−10.5	7	−51,103.5	34,069
WI	4,992	0.5	4	2,496	19,968
WV	1,793	3.5	2	6,275.5	3,586
WY	454	−5.5	3	−2,497	1,362

10. Based on the data from the chart, centroid C_1 $(\bar{x}, \bar{y})$ for the 48 states is approximately (–0.37, 1.92).

11. The adjustment for Alaska and Hawaii is a small shift of the population center to the west. Although each state is located far from the rest of the country (therefore, the coordinate values are very large), their populations are small. A slight shift to the west would suggest (–0.5, 2) from the calculated centroid. Demonstrate to students how the value of $P_i x_i$ (and $P_i y_i$) becomes a larger value for Alaska and Hawaii than for other states with small population figures.

12. The population center determined by the Bureau of the Census is approximately C (–0.5, 1.7) given the coordinate system developed for the previous problems. This point averages in all states. If this point is considered the actual population center, then the centroid calculated in problem 11 is very close to the actual population center.

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10. Using the data from your 1990 State Population Chart, estimate a population center by calculating the centroid of the set of points. The centroid C_1 $(\bar{x}, \bar{y})$ for the 48 states discussed would be based on the following summations:

$$\bar{x} = \frac{\sum_{i=1}^{48} P_i x_i}{\sum_{i=1}^{48} P_i} \text{ and } \bar{y} = \frac{\sum_{i=1}^{48} P_i x_i}{\sum_{i=1}^{48} P_i}$$

11. Locate the centroid on the United States map and label it as C_1. Adjust this point to include your estimate of including Alaska and Hawaii as discussed in problem 9. (Remember, Hawaii and Alaska are particularly important small states in the location of the country's population center.)

12. The location of the 1990 population center calculated by the Bureau of the Census is published in a number of sources, including special publications from the U.S. Department of Commerce, Bureau of the Census. The *Statistical Abstract of the United States* is an excellent source for comparing the 1990 center with previous centers. The population center as determined by the Bureau of the Census will be shared with you by your teacher. Compare your estimate with the center derived from the Bureau of the Census.

The Smaller Picture

Another method to consider will be discussed as the "smaller picture" of the population data. The "Belgium problem" indicated how the magnitude of the larger communities determined the location of the population center. Consider the following process in estimating a population center for the 48 connected states:

Step 1: Again construct an *xy*-coordinate system on the map of the United States.

Step 2: Estimate a population center for each of the 10 states with the greatest population. Here again, place a dot at the location of each state's capital.

13. The values below are obtained from the population data sheet and the coordinate values used in the "The Big Picture" method.

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13. Based on the ten population centers placed on the map, copy and complete the data sheet that follows.

Data for Determining the U.S. Center of Population "The Smaller Picture"

State (abbreviation)	Population P_i (in thousands)	x-coordinate value	y-coordinate value	P_ix_i	P_iy_i
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___

14. Determine a centroid C_2 based on the ten points listed in the table.

$$\bar{x} = \frac{\sum_{i=1}^{10} P_i x_i}{\sum_{i=1}^{10} P_i} \text{ and } \bar{y} = \frac{\sum_{i=1}^{10} P_i y_i}{\sum_{i=1}^{10} P_i}$$

15. Locate this centroid on the United States map. Again, consider adjusting this point to include Alaska and Hawaii as summarized in problem 9. How does C_2 compare to C_1?

Practice and Applications

16. The census populations for 1980, 1970, and 1960 are also included on the Population Data Sheet. Using either the "Bigger Picture" or the "Smaller Picture" method, determine a population center for one of these years. Compare your center to the center plotted on the map presented at the beginning of this lesson.

17. Some "futurists" speculate on what the population of countries in North America will look like 50 to 100 years from now. Two extreme scenarios were developed. One scenario

Data for Determining the U.S. Center of Population "The Smaller Picture"

State (abbreviation)	Population P_i (in thousands)	x-coordinate values	y-coordinate values	P_ix_i	P_iy_i
CA	29760	−11.5	2.5	−342240	74400
NY	17990	5.5	4.5	98945	80955
TX	16987	−3.5	−2.5	−59454.5	−42467.5
FL	12938	3.5	−2	45283	−25876
PA	11882	5	3.5	59410	41587
IL	11431	0.5	2.5	5715.5	28577.5
OH	10847	3	2.5	32541	27117.5
MI	9295	2	4	18590	37180
NJ	7730	6	3.25	46380	25122.5
NC	6629	5	1	33145	6629

14. Centroid C_2 based on the ten points involved in the table is

$\bar{x} = -0.5$ and $\bar{y} = 1.9$.

15. Both points are very close. Plotting the points on a map would illustrate this. An adjustment for Alaska and Hawaii would shift this location slightly to the left. One approximation might be (−0.7, 1.8). This point would also be very close to the center described by the Bureau of the Census (−0.5, 1.7).

Practice and Applications

16. This can be an exciting computer application problem. A spreadsheet allows students to paste the other population values over the 1990 column and watch the resulting changes in the centroid. Students could also develop their own formulas to calculate the other centroid values. The possibilities to expand this problem into other activities or miniprojects are obvious. The following summaries were obtained from the spreadsheet mentioned:

Decade	Estimated Centroid from Population Data Sheet	Estimate of Census Bureau's Population Center
1960	(0.55, 2.28)	(0.5, 2.0)
1970	(0.37, 2.24)	(0.4, 2.0)
1980	(1.06, 2.10)	(–0.2, 1.5)

17. Both predictions are based on extreme ideas. Locating a population near the Canadian border was actually suggested by an environmental group that felt the effects of global warming would make at least one-third of the United States desert. The second prediction that would place the population center in the southwest corner assumes the general trend of people to the south and west would continue. Also, migration of citizens from Mexico into this country would add to this shift.

18. Finding a "center" for a globe is difficult to imagine. The matching of a location to a point that was observed in a two-dimensional map is not possible on a globe or three-dimensional model. This makes the meaning of a center very difficult to explain.

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suggests that the population center of the United States will be located close to the Canadian border. Another scenario suggests the population center will be located in the southwestern corner of the United States. Explain what would cause a population center to be located in either of these two extremes.

18. Examine a globe of the world. If you were interested in determining the population center of the world, what new questions must be addressed? Do you think a population center for the world exists? Is it meaningful? Explain.

Population Data Sets of the United States
Compiled from the *Statistical Abstract of the United States* (1991)

State	Population 1960 (in thousands)	Population 1970 (in thousands)	Population 1980 (in thousands)	Population 1990 (in thousands)
AK	226	303	402	550
AL	3267	3444	3894	4041
AR	1786	1923	2286	2351
AZ	1302	1775	2718	3665
CA	15717	19971	23668	29760
CO	1754	2210	2890	3294
CT	2535	3032	3108	3287
DE	446	548	594	666
FL	4952	6791	9746	12938
GA	3943	4588	5463	6478
HI	633	770	965	1108
IA	2758	2825	2914	2777
ID	667	713	944	1007
IL	10081	11110	11494	11431
IN	4662	5195	5490	5544
KS	2179	2249	2364	2478
KY	3038	3221	3661	3685
LA	3257	3645	4206	4220
MA	5149	5689	5737	6016
MD	3101	3924	4217	4781
ME	969	994	1125	1228
MI	7823	8882	9262	9295
MN	3414	3806	4076	4375
MO	4320	4678	4917	5117
MS	2178	2217	2521	2573
MT	647	694	787	799
NC	4556	5084	5882	6629
ND	632	618	653	639
NE	1411	1485	1570	1578
NH	607	738	921	1109
NJ	6067	7171	7365	7730
NM	951	1017	1303	1515
NV	285	489	800	1202

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19. Make certain students have developed a good cutout of the country. The map on *Activity Sheet 8* is an appropriate size to accomplish this. Allow students to experiment with the placement of the raisins. The following illustration will balance at a point close to the population center discussed in this lesson.

NY	16782	18241	17558	17990
OH	9706	10657	10761	10847
OK	2328	2559	3025	3146
OR	1769	2092	2633	2842
PA	11319	11801	11864	11882
RI	859	950	947	1003
SC	2383	2591	3122	3487
SD	681	666	691	696
TN	3567	3926	4591	4877
TX	9580	11199	14229	16987
UT	891	1059	1461	1723
VA	3967	4651	5347	6187
VT	390	445	511	563
WA	2853	3413	4132	4867
WI	3952	4418	4706	4992
WV	1860	1744	1950	1793
WY	330	332	470	454

Balancing the Map (Optional)

19. Consider estimating the population center with a map and eight raisins. Copy a map of the United States on poster paper.

Step 1: Cut out the map of the 48 connected states. Cut along the coast lines and the Canadian and Mexican borders.

Step 2: Tape eight raisins to eight estimated population centers for the country. Let each raisin represent approximately 30 million people. Based on the populations provided for each state, tape the eight raisins at the locations you think represent a population center for approximately 30 million people.

Step 3: Use the blunt end of the pencil discussed in previous lessons to balance your shape with the raisins.

Step 4: Find a balance point and record it. How does the balance point compare to the points calculated from the table?

UNIT V

Minimizing Distances by a Center

LESSON 11

Minimizing Distances on a Number Line

Materials: No materials are required for this lesson.
Technology: scientific calculator
Pacing: 1 to 2 class sessions

Overview

Lessons 11, 12, 13, and 14 address other notions of center. This lesson deals with the problems of finding a center that minimizes distances traveled to several points along a number line. The median location turns out to be the center needed to solve the problem. As indicated in the earlier lessons of this module, the mean and the median become primary summaries of data sets. Also similar to the development of the mean center, the following series of lessons begin with a one-dimensional application and expands to two dimensions in the concluding lessons.

Teaching Notes

Students are invited to estimate the best location for the center in each of the settings and then to try various locations and see which location works best. Proofs that the solutions provided are optimal are not given as they are tedious, difficult, and do not provide additional insight beyond that gained through exploration. This lesson deals with the center through experimentation and conjecture. Students are expected to observe the median as the estimate of the location to minimize the distances traveled in the problems presented.

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LESSON 11

Minimizing Distances on a Number Line

What are some of the things a person should consider when choosing a location for a distribution center that trucks supplies to several different stores?

Does the number of stores supplied by the warehouse matter?

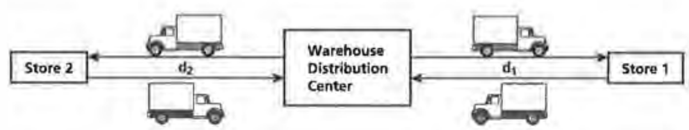
Suppose it were your job to load a truck with supplies at a warehouse, drive the truck to a store, unload it, drive back to the warehouse, reload the truck with more supplies, drive it to a second store, unload it, and return to the warehouse. If you could choose the site for the warehouse, where would you put it?

The location of a center in this problem is different from location of a center in the problems involving a balance point or a population center. You will undoubtedly want to put the warehouse in a central location, but what is meant by "central" in this case? The goal is to find a location that minimizes the total distances driven to the stores and back.

OBJECTIVES

Determine the location along a number line that minimizes the sum of the distances to selected points.

Use the median of a set of points as an estimate for selecting a point to minimize the sum of distances.



Solution Key

Discussion and Practice

1. Answers will vary. The triangle inequality implies that the warehouse should be somewhere along the line connecting the stores. However, some students will simply want to note that the shortest distance between two points is along a straight line.

2. Answers will vary. Students might anticipate that the midpoint of the line between the two stores works well.

3.

Location of Warehouse	Distance to Store 1	Distance to Store 2	Sum of the Distances
Example:			
8	5	1	6
6	3	1	4
4	1	3	4
3	0	4	4
1	2	6	8

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INVESTIGATE

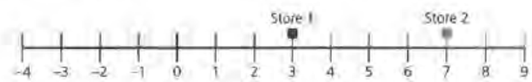
Location, Location, Location

Call the distance from the warehouse (no matter where it is located) to store 1 in the diagram d_1 and the distance from the warehouse to store 2 d_2. The total distance traveled by the truck in this problem is $2d_1 + 2d_2$. Why?

The goal in this problem is to *minimize* this *sum of the distances* $2d_1 + 2d_2$. This problem can be simplified by first minimizing $d_1 + d_2$. (The location of the warehouse that minimizes the sum of twice each distance is the same as the location that minimizes the sum of the distances—it does not matter if several one-way trips or several round trips are considered in this problem.)

Discussion and Practice

To find this new "center," visualize the setup of this problem on a number line (similar to the way the previous investigation of a center of balance was started). Each store will be placed on the number line. The goal of the problem is to determine the location of a warehouse that minimizes the sum of the distances discussed in this problem. To start this discussion, suppose the stores are located along a straight road in which the location of store 1 is 3 and the location of store 2 is 7.



1. The best location for the warehouse is somewhere along the line connecting the two stores. Why?
2. What would be your initial estimate for the location of a warehouse that minimizes the sum discussed in this lesson? Why did you choose that location?
3. Consider locating the warehouse at each of the locations designated in the following chart. Determine the sum of the one-way distances to each of the stores based on the warehouse location:

4. Consider the results from the chart.

 a. Locations 3, 4, and 6

 b. Any location between the stores or including the stores

5. The center of balance is the mean, which is one of the locations that minimizes the sum of the distances for this problem.

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Location of Warehouse	Distance to Store 1	Distance to Store 2	Sum of the Distances
Example:			
8	5	1	6
6	_____	_____	_____
4	_____	_____	_____
3	_____	_____	_____
1	_____	_____	_____

4. Consider the results from the chart.
 a. What location or locations minimize the sum of the distances?
 b. How would you describe the possible locations of the warehouse?

5. Would the location that minimizes the sum of the distances for the two stores be the same location if the center of balance were used to locate the warehouse. Why or why not?

The location of the warehouse for this first example is simplified by the fact that only two stores are considered. Consider an expansion to this problem by placing a third store at location −3 on this number line.

The location of the warehouse must minimize the sum of the distances:

distance from warehouse to Store 1
+ distance from warehouse to Store 2
+ distance from warehouse to Store 3.

6. Consider several possible locations to place this warehouse. Each of the possible locations are identified in the following diagram.

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6. a. See below.

a. Copy and complete the chart summarizing the distances discussed in this problem:

Location of Warehouse	Distance to Store 1	Distance to Store 2	Distance to Store 3	Sum of the Distances
Example:				
−4	1	7	11	19
−3	___	___	___	___
0	___	___	___	___
3	___	___	___	___
5	___	___	___	___
7	___	___	___	___
8	___	___	___	___

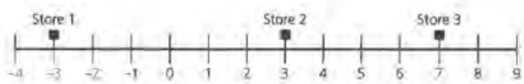
b. Would you have added any other possible locations of the warehouse to this chart? If yes, determine the sum of the distances for the additional locations you considered.

7. Based on the information summarized by the table in question 6, what location or locations would minimize the sum of the distances?

8. How would you describe the location of the warehouse that minimizes the sum of the distances?

9. Would the location that minimizes the sum of the distances for three stores be the same location if the center of balance were used to locate the warehouse? Why or why not?

10. Suppose store 3 had been at location 9, rather than at location 7.



Location of Warehouse	Distance to Store 1	Distance to Store 2	Distance to Store 3	Sum of the Distances
Example:				
−4	1	7	11	19
−3	0	6	10	16
0	3	3	7	13
3	6	0	4	10
5	8	2	2	12
7	10	4	0	14
8	11	5	1	17

b. Students may add locations to the list, however, the list above includes the best location (location 3).

7. Location 3, the location of store 2, minimizes the sum of the distances.

8. The median of the locations of the stores minimizes the sum of the distances.

9. No. The mean and the median are not the same.

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10. a. See below.

a. Copy and complete the chart summarizing the distances:

Location of Warehouse	Distance to Store 1	Distance to Store 2	Distance to Store 3	Sum of the Distances
−3	____	____	____	____
0	____	____	____	____
3	____	____	____	____
5	____	____	____	____
7	____	____	____	____
9	____	____	____	____

b. Would you have added any other possible locations of the warehouse to this chart? If yes, determine the sum of the distances for the additional locations you considered.

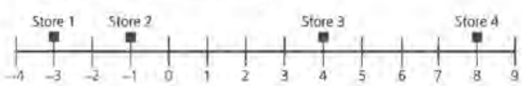
c. What is the best location for the warehouse? In what way, if any, does moving store 3 from location 7 to location 9 affect the best location for the warehouse? Why?

SUMMARY

Different types of centers are based on different types of problems. A central location designed to minimize the sum of the distances to selected points along a number line is estimated by the location of the median of a set of points. Depending on the number of points described in the problem, the median may not necessarily be the only location that minimizes the sum of the distances, however, it is the estimate that begins the investigation of this location.

Practice and Applications

11. Consider four stores located along a number line as illustrated in the following diagram:



a. Based on your work with two and three stores, where would you suggest a warehouse be located to minimize the sum of the distances?

Location of Warehouse	Distance to Store 1	Distance to Store 2	Distance to Store 3	Sum of the Distances
−3	0	6	12	18
0	3	3	9	15
3	6	0	6	12
5	8	2	4	14
7	10	4	2	16
9	12	6	0	18

b. Students may add locations to the list, however, the list above includes the best location (location 3).

c. The median is the best location. Moving store 3 from location 7 to location 9 does not affect the median, so it does not affect the best location for the warehouse.

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b. Determine the sum of the distances for at least five possible locations of the warehouse. Include in your locations the one you think is the best. Organize the results of each location in a chart similar to the one below.

Location of Warehouse	Distance to Store 1	Distance to Store 2	Distance to Store 3	Distance to Store 4	Sum of the Distances
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___
___	___	___	___	___	___

12. Describe the best location or locations of the warehouse for four stores located along a number line.

13. If it were necessary to locate the warehouse at the location of an actual store, what store or stores would you choose in problem 12? Why?

14. Why is it more difficult selecting the location of a warehouse for four stores than for three stores?

15. Draw a number line and place the location of five stores along the number line.

a. Where would you locate a warehouse to minimize the sum of the distances to the stores?

b. Is there more than one possible location for a warehouse when there are five stores? Why or why not?

Practice and Applications

11. a. The median will be the best location.

b. Answers will vary. Here is a good set of possible locations.

Location of Warehouse	Distance to Store 1	Distance to Store 2	Distance to Store 3	Distance to Store 4	Sum of the Distances
–1	2	0	5	9	16
2	5	3	2	6	16
4	7	5	0	4	16
6	9	7	2	2	20
8	11	9	4	0	2

12. The warehouse could be put at –1, at 4, or at any location between –1 and 4.

13. Store 2 or store 3 would work, as they are at locations described in the solutions to problem 12.

14. There are more points to consider. Because the median is the average of two values here, there is more than one location that minimizes the sum of the distances. The median works, but so does any location between the two stores used to calculate the median.

15. **a.** Answers will vary. In each case, however, the median will be the best location for the warehouse.

b. No. Since 5 is an odd number, the median is the same as the location of one of the stores and provides the unique solution.

LESSON 12

Taxicab Geometry

Materials: *Activity Sheet 13*
Technology: scientific calculator
Pacing: 1 to 2 class sessions

Overview

This lesson expands the description of the problem started in Lesson 11 to a two-dimensional application. The problems posed to the student require a minimization of the distances traveled to several locations on a grid. The distances are measured in units representing only up/down or left/right units (i.e., taxicab units). Locations are conjectured and then tested. From the work in Lesson 11, students are expected to estimate the requested centers by using the median values of the locations. This median center must be determined using both the x- and y-locations of the warehouses discussed in the problems. The median location provides the desired minimization of traveled distances and suggests to the students a generalization of this investigation.

Teaching Notes

Charts are carefully organized to encourage students to investigate a location and record the resulting distances. The organization of the charts was designed to guide the students into expanding the conclusions derived from the previous lesson. The lesson ends with students seeing a weighted value to one of the warehouse locations. The weighted value represents a location of a warehouse that is traveled to more than once. The difference between the interpretation of a weighted location in this lesson is contrasted to the weighted value of a location in the mean center applications.

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LESSON 12

Taxicab Geometry

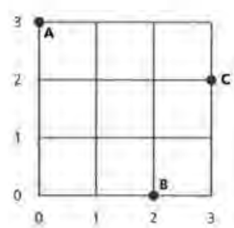
A warehouse is to be located so that it can supply three stores, labeled A, B, and C in the diagram below. If the warehouse were to be added to an existing store, which store would you select?

What other in-town location would be a better site? Why?

Suppose there are three stores located at the intersections of streets in a town. A street map of the town shown below looks like a grid. All of the streets run either north-south or east-west and are evenly spaced. Locations of the stores can be designated by using (x, y) coordinates. For example, store B is located on this map as location (2, 0). What is the location of store A?

OBJECTIVE

Determine the location on a two-dimensional grid that minimizes the sum of the distances to selected points.



INVESTIGATE

Who's Minding the Store?

Where is the best place to put a "central" warehouse? The goal is to *minimize* the sum of the distances from the warehouse to the stores. Measure this distance by counting the number of blocks traveled.

Solution Key

Discussion and Practice

1. 3 units.
2. No. Any movement of A that keeps it on the grid will make it closer to B than it is now.
3. Answers will vary.
4. The distance from the warehouse to store B is 1 unit. The distance from the warehouse to store C is 2 units.

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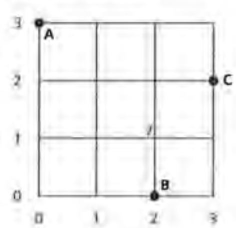
Discussion and Practice

1. What is the distance from store B to store C in the diagram?
2. Store C could be moved to a different location and remain the same distance from store B as it is now. Could store A be moved to a different location on the grid and have it be the same distance from store B as it is now?
3. What is your initial estimate for the location of a warehouse that minimizes the sum of distances to the three stores? Why did you choose that location?
4. Consider putting the warehouse discussed in problem 3 at the location with coordinates (2, 1). The distance from this warehouse to store A is 4 blocks. Determine the distance from the warehouse to store B and from the warehouse to store C.
5. Consider locating the warehouse at each of the locations designated in the following chart. Find the distances needed to fill in the table and find the sum of the distances.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances
Example:				
(2, 1)	4	___	___	7
(3, 1)	___	___	___	___
(2, 2)	___	___	___	___
(2, 3)	___	___	___	___

If the warehouse is located at (2, 1), the sum of the distances is 7 blocks. Label location (2, 1) with a "7" on the map to designate this location.



5.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances
Example:				
(2, 1)	4	1	2	7
(3, 1)	5	2	1	8
(2, 2)	3	2	1	6
(2, 3)	2	3	2	7

6. a. 5 + 2 + 1 = 8

b. 3 + 2 + 1 = 6

7. (2, 2)

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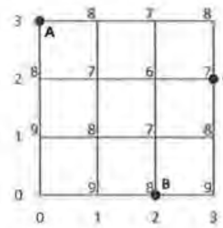
6. Consider locating the warehouse at other locations.

   a. If the warehouse is located at (3, 1), what is the sum of the distances to the three stores?

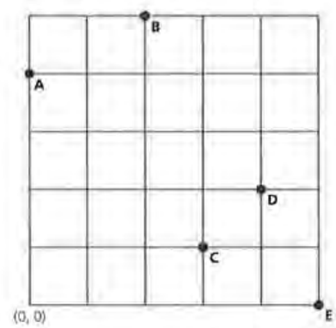
   b. If the warehouse is located at (2, 2), what is the sum of the distances?

Placing a number representing the sum of the distances at each coordinate location gives the following summary:



7. According to this summary, what location minimizes the sum of the distances?

Expand this problem to include five stores located as shown on the grid.



The question remains to determine the location of a warehouse that minimizes the sum of the distances to each store. One possible location for the warehouse is (4, 1), included on the following table. Another possible location is (4, 2). This location

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8. Answers will vary. One possible chart is shown below.

indicates the possibility of locating the warehouse at the same location as store D.

8. In addition to the locations of (4, 1) and (4, 2), choose four other possible locations for the warehouse. For each local, calculate the sum of the distances to the stores. Copy and complete the following table for each of the choices:

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Distance to Store D	Distance to Store E	Sum of the Distances
Example:						
(4, 1)	7	6	1	1	2	17
(4, 2)	6	5	2	0	3	16
____	____	____	____	____	____	____
____	____	____	____	____	____	____
____	____	____	____	____	____	____
____	____	____	____	____	____	____

9. What is the best location for the warehouse as represented with the above locations?

10. What is the x-coordinate of the best location for the warehouse? How is this related to the x-coordinates of the locations of the five stores?

11. What is the y-coordinate of the best location for the warehouse? How is this related to the y-coordinates of the locations of the five stores?

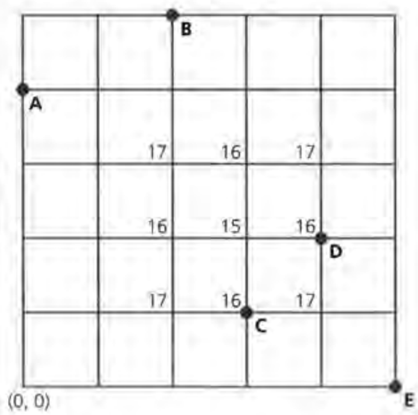
12. Based on the solutions to problems 10 and 11, how would you describe the location for the warehouse that minimizes the sum of the distances?

SUMMARY

The location that minimizes the sum of distances to selected points along a grid is given by the **median** of the x-coordinates and the **median** of the y-coordinates. Depending on the number of points in the problem, there may be more than one location that minimizes the sum of the distances.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Distance to Store D	Distance to Store E	Sum of the Distances
Example:						
(4, 1)	7	6	1	1	2	17
(4, 2)	6	5	2	0	3	16
(2, 3)	3	2	3	3	6	17
(3, 3)	4	3	2	2	5	16
(3, 2)	5	4	1	1	4	15
(2, 4)	2	1	4	4	7	18

9. The best location is (3, 2), just above point C and to the left of point D. The sum of the distances from this location is 15.



10. The best x-value is 3, the median of the 5 x-values of the stores.

11. The best y-value is 2, the median of the 5 y-values of the stores.

12. The best location is the point given by (median x, median y)

Practice and Applications

13. a. Answers will vary. There is an even number of stores, so there could be more than one optimal value for the x-coordinate; likewise for the y-coordinate.

b. Answers will vary. Here is one possible set of locations. One optimal location is (2, 2).

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Distance to Store D	Sum of the Distances
(0, 2)	0	2	6	4	12
(1, 2)	1	1	5	3	10
(2, 2)	2	2	4	2	10
(3, 3)	4	2	2	2	10
(3, 1)	4	4	4	0	12
(4, 2)	4	4	2	2	12

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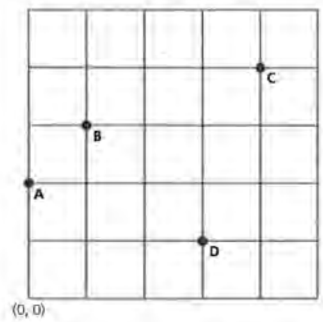
Practice and Applications

13. Suppose there are four stores, located as on the grid below.

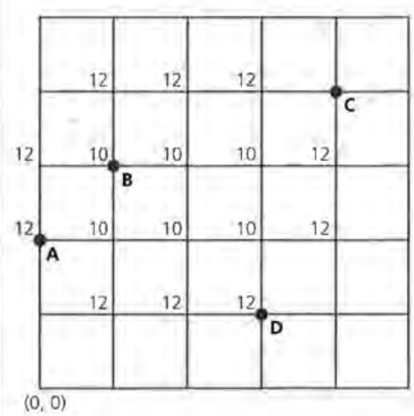
a. Based on your work with three stores and with five stores, what is the best location for the warehouse? Do you think there might be more than one location that is "best"?

b. Choose possible locations for the warehouse, calculate the sum of the distances for each location, and fill in these values on the table.



Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Distance to Store D	Sum of the Distances

14. Consider the first example (with three stores) again. Suppose that you need to go from the warehouse to store A once, from the warehouse to store B once, but from the warehouse to store C three times. How does this change the problem? Now where is the best place for the warehouse?



14. We can think of this as a problem with five points, where the location of C is the location of three of the five points. The best location for the warehouse is at (median of *x*, median of *y*). The median of the five *x*-coordinates is the *x*-coordinate of point C, which is 3, and the median of the five *y*-coordinates is the *y*-coordinate of point C, which is 2. Thus, (3, 2) is the solution. This is like having 5 stores. The best location is (median *x*, median *y*), which is (3, 2), the location of store C.

LESSON 13

Helicopter Geometry

Materials: rulers, *Activity Sheet 13*
Technology: A scientific calculator or graphing calculator will do, but a geometry software package such as Geometer's SketchPad or Cabri is preferable.
Pacing: 2 class sessions

Overview

This lesson (as well as Lesson 14) poses some very challenging problems. Expanding the "taxicab" geometry to "helicopter" geometry simply indicates the distances are measured by direct line segments from one location to another. The challenge of finding the location of a center to minimize the sum of the distances is now more complex. Although a median point provides an estimate to the questions, it is not the complete picture.

Problems are developed guiding students through the process of selecting an approximate center for three points, four points, and five points. Even with a limited number of points, however, the work involved is tedious. For this reason, it is recommended students explore this problem using a computer application such as Cabri or Geometer's Sketchpad. With or without software, students use the same approach developed with the taxicab lesson—choose a point, find the sum of the distances from that point, and then make new decisions based on the results. The students are provided points, guided through the calculations, and directed to base their selections accordingly. The software sited will make this process easier. "Centers" can easily be evaluated as they are moved around the grid.

Teaching Notes

The development of the prepared lesson is designed for the situation in which access to computer software is limited. The specific problems presented in the student materials are arranged to guide the students through the process discussed in the Overview section. Included in this teacher edition, however, is an extensive description of developing similar problems using the Geometer's Sketchpad. The explanation provides suggestions on how to set up the problems. Details related to using the software are not developed, however, as that information constantly changes and is not the important aspect of this lesson.

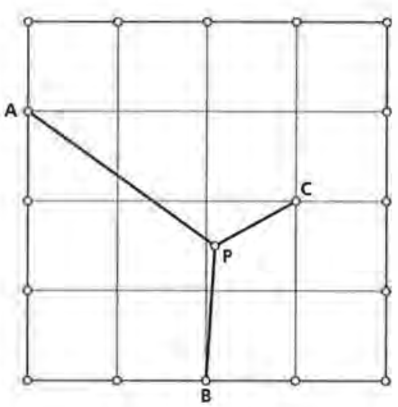
If the problems are developed using the recommended software, one class period will be needed to allow students to feel comfortable with the software. This time is well-spent, however, as the remaining work with the lesson will be more productive as students will not be required to set up each of the calculations needed to evaluate the centers.

Technology

Technology is key to making this lesson meaningful. As the problems are presented and the strategy for deriving an estimate are tedious, use of technology is considered essential. This lesson is most effectively presented using a geometry software package. The Geometer's Sketchpad and Cabri were both tested with the problems presented in the lesson. Each package offered extensions and insights to the objective of the lesson. If this level of technology is not available, a scientific or graphing calculator is needed to complete the calculations.

The Geometer's Sketchpad or similar software is a tremendous tool for understanding the insights of this lesson. If you have the Geometer's Sketchpad you can let your students explore this problem using the computer. Even without software, students can use the same approach used with the taxicab geometry. That is, have them choose a point and find the sum of the distances for that point. Then label that point with the value found for the sum of the distances. After doing this for several points, they will have built up part of a contour plot. By looking at this partial contour plot they should be able to see where to look for the best warehouse location.

We recommend, however, you approach the problems differently if you are going to use software. In the student materials, a coordinate grid was developed so that the locations and estimates of the centers could be compared and checked. If you are interested in designing a grid and using the specific points of the exercises, your first task would be to construct segments that form a coordinate grid (as illustrated with the examples). The goal of this lesson, however, is to develop an understanding of the location of the center, and this does not require a coordinate grid.



Length (Segment PA) = 126.491 pixels

Length (Segment PB) = 77.162 pixels

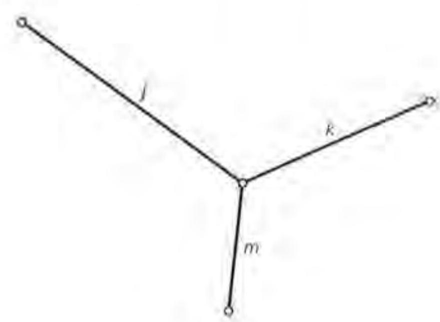
Length (Segment PC) = 50.537 pixels

Length (Segment PA) + Length (Segment PB) + Length (Segment PC) = 254.190 pixels

(Length (Segment PA) + Length (Segment PB) + Length (Segment PC))/5 = 50.84 pixels

Working with Three Points (Problems 1–11)
To use the Sketchpad, plot any three points and locate a fourth point that will serve as the center; label this as point *P*. Form the segments from this point to each of the points *A*, *B*, *C*; measure the length of these segments. With the Calculate option of Sketchpad, find the sum of the lengths. Now let students move the center point; challenge them to determine the location of the point that minimizes the sum of the distances. An example is included. (Note: The measure of the distances can be in inches, centimeters, or pixels.)

Appropriate location of this center will suggest to students that the minimum sum of the distances is determined by a center point that forms 120° central angles. This is the main insight to be gained from this section. After the students have observed this feature, add the angle measures to the list of calculations developed for this experiment. Students will now be able to determine the center by minimizing the sum of the distances or by locating a point that forms center angles of 120°.



Note: The proof of the 120° result is complicated and uses higher mathematics.

Length (Segment j) = 134.807 pixels

Length (Segment k) = 99.539 pixels

Length (Segment m) = 63.506 pixels

Length (Segment j) + Length (Segment k) + Length (Segment m) = 297.852 pixels

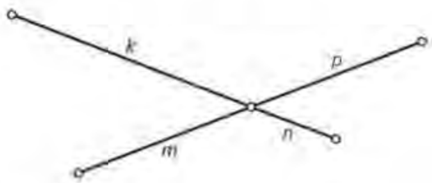
Angle (ADC) = 120.228°

Angle (ADB) = 120.092°

Working with Four Points (Problems 12–21)

The contour plot idea described above can be used here as well. Have students try a few points, combine their answers on a single graph, and then, by looking at the graph, try other points until they find the minimum.

Location of the center point for four points is also rather easily developed with the Sketchpad. Again, plot four points, labeled *A, B, C,* and *D* and build a measure of the lengths from a center point *P.* Move the center point until the sum of the distances is a minimum. This location should be close to the location of the intersection of the diagonals as illustrated below.



Challenge students to check this summary with the tools of the Sketchpad.

Length (Segment k) = 169.000 pixels

Length (Segment m) = 120.934 pixels

Length (Segment n) = 57.940 pixels

Length (Segment p) = 119.507 pixels

Length (Segment k) + Length (Segment m) +
Length (Segment n) + ... = 467.381 pixels

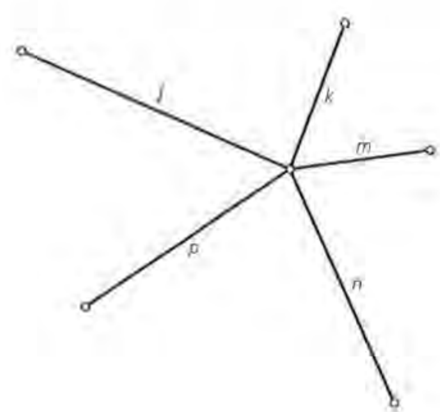
Note: These "minimize the sum of the distances" problems are similar to—but different from—the so-called "minimum network" problems. A minimum network problem with four points is one in which the goal is to draw a network that touches each of the four points and that has the shortest total length (i.e., if the network is drawn with a pen the minimum network uses the least amount of ink necessary). For example, here is a sketch of the solution to a four-point minimum network problem:

The two points at which the network has branches are known as **Steiner Points.** Although minimum network problems are similar to minimum total distance problems, they lead to different solutions.

Working with Five or More Points

The location of five points begins to expand and stretch students' understanding of this material. Technically, the best location for the warehouse is at what is called the "spatial median." There is no closed form solution to finding the spatial median for a general number of points. (Rather, one must program a computer with a version of something called the Newton-Raphson Method, which is itself a multidimensional version of Newton's Method for finding the zeros of a function. This involves lots of calculus and goes *far* beyond the level of high school mathematics.) Nonetheless, students can use trial and error to find an approximate solution. The general rules developed above should help them find good starting values for their trial-and-error measurements.

The Sketchpad or other software will be a real blessing if you want to explore what happens with five or more points. To appropriately use the Sketchpad, encourage students to first select five points that if connected form a convex pentagon. (Concave examples should be treated separately.) Estimate a center, P, and determine the sum of the distances from P to the five points. Move P until the sum is a minimum. Mark this point!



Length (Segment j) = 193.830 pixels

Length (Segment k) = 104.403 pixels

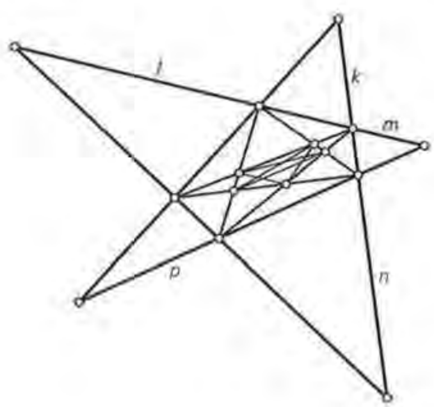
Length (Segment m) = 92.914 pixels

Length (Segment n) = 169.664 pixels

Length (Segment p) = 162.542 pixels

Length (Segment j) + Length (Segment k) +
Length (Segment m) + ... + ... = 723.353 pixels

You might want to further challenge students by connecting the five points to form a "star." The points of intersection within the star yield a second set of points that form a concave pentagon. Create a star with these five points, and again, observe that another pentagon is formed. If this process is continued, the subsequent stars will "converge" on the center point. Make sure this observation is verified with the Sketchpad.



Length (Segment j) = 180.677 pixels

Length (Segment k) = 111.364 pixels

Length (Segment m) = 107.912 pixels

Length (Segment n) = 175.419 pixels

Length (Segment p) = 149.807 pixels

Length (Segment j) + Length (Segment k) +
Length (Segment m) + ... + ... = 725.179 pixels

Encourage students to develop a *concave* pentagon and estimate how to determine its center. Obviously the approach of forming a star will not work and, therefore, another approach must be developed. In this example, developing the sum of the distances from an estimated center point and then moving that point to minimize distances should be considered.

This should also be the method developed or emphasized if a problem involving six or more points is attempted.

It is possible to consider giving different weights to the various points. For example, if there are three stores but one of them must be visited twice, then the situation is the same as that of the four-point problem, with one of the stores counting as two of the points.

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LESSON 13

Helicopter Geometry

What would be the advantages of using a helicopter instead of a truck to transport supplies from a distribution center to area stores?

What would the disadvantages be?

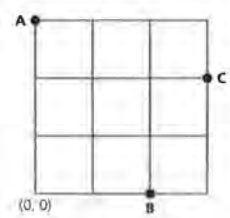
OBJECTIVE

Determine the location on the plane that minimizes the sum of distances to selected points.

In Lesson 12, the best location for a warehouse that served a group of stores was found under the condition that travel was restricted along gridlines. The "center" turned out to involve the medians of the x- and y-coordinates of the stores. Suppose there is no restriction that travel must be along gridlines. Instead, suppose that you have a helicopter and can travel on a direct line to each point. It turns out that this makes the solution—the best location for the warehouse—harder to find!

INVESTIGATE

Consider three stores given by the points on the map shown below. The goal of the next set of problems is to find the best location for the warehouse, given that you can fly directly from the warehouse to each point. In other, words, *the warehouse does not have to be at the intersection of streets any more!* Use the gridlines as a kind of graph paper to help keep track of locations on the map.



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Solution Key

Discussion and Practice

1. 2 units
2. Point C has coordinates (3, 2), so the distance formula gives

   $\sqrt{(3-0)^2+(2-0)^2} = \sqrt{13}$ units.
3. Answers will vary.
4. The distance from the warehouse to store B is

   $\sqrt{(2-1)^2+(0-1)^2} = \sqrt{2}$ units.

   units. The distance from the warehouse to store C is

   $\sqrt{(3-1)^2+(2-1)^2} = \sqrt{5}$ units.
5. (See table below.)

Discussion and Practice

If the warehouse is placed at (0, 0), then the distance from the warehouse to point A is 3 units.

1. What is the distance from (0, 0) to point B?
2. Use the Pythagorean Theorem to verify that the distance from (0, 0) to point C is $\sqrt{3^2+2^2} = \sqrt{13}$ units.
3. What is your initial estimate for the location of a warehouse that minimizes the sum of distances to the three stores? Why did you choose that location?

Consider Three Points

If the warehouse is at location (0, 0), then the sum of the distances is $\sqrt{3^2+2^2} = \sqrt{13}$, or about 3.6.

4. Locate the warehouse at location (1, 1). The distance from the warehouse to store A is $\sqrt{5}$ units (again, derived by way of the Pythagorean Theorem). Determine the distance from the warehouse to store B and from the warehouse to store C.
5. Consider locating the warehouse at each of the locations designated in the following chart. Find the distances needed to complete the table and find the sum of the distances.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances (rounded to one decimal place)
(0, 0)	3	____	$\sqrt{13}$	8.6
(1, 1)	$\sqrt{5}$	____	____	____
(1, 2)	____	____	____	____
(2, 2)	____	____	____	____

Before a decision can be made concerning the location of the warehouse, locations that are not at the intersection of gridlines must also be considered. For example, suppose the warehouse is located at (1.5, 1.5). One way to estimate the distances involved is to measure the distance to each point representing a store with a piece of string. Anchor one end of the string at one of the possible locations of the warehouse; stretch the string and mark the location to store A. Again, anchor the string at the warehouse location and mark the distance to store B. Finally, anchor the string and mark the distance to store C. The accumulated distances of the string represents the sum of the distances requested in this problem.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances (Rounded to one decimal place)
(0, 0)	3	2	$\sqrt{13}$	8.6
(1, 1)	$\sqrt{5}$	$\sqrt{2}$	$\sqrt{5}$	5.9
(1, 2)	$\sqrt{2}$	$\sqrt{5}$	2	5.7
(2, 2)	$\sqrt{5}$	2	1	5.2

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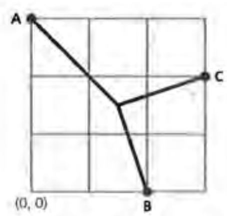
6. (See table below.)

In order to improve these measurements, plot the stores A, B, and C, on the larger grid provided with this material. Express the final distance of the string in terms of the units of this grid. For example, if you place the warehouse at (1.5, 1.5), then the distance to point A is estimated at 2.1 units; similarly, the distance to B is 1.6 units, and the distance to C is 1.6 units. The sum of the distances is approximately 5.3 units. A more accurate method to investigate this problem is to use one of several software packages such as The Geometer's SketchPad or Cabri. Each application allows you to locate the stores on a grid and develop a sum of the distances.



6. Consider locating the warehouse at each of the locations designated in the following chart. Use a piece of string to find the distances to each of the stores and record those in the table. Then find the sum of the distances.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances (rounded to one decimal place)
(1.5, 1.5)	2.1 units	1.6 units	1.6 units	5.3 units
(1.5, 2)	______	______	______	______
(2.5, 2.5)	______	______	______	______

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances (Rounded to one decimal place)
(1.5, 1.5)	2.1	1.6	1.6	5.3
(1.5, 2)	1.8	2.1	1.5	5.4
(2.5, 2.5)	2.5	2.5	0.7	5.7

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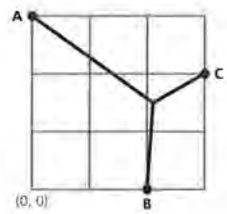
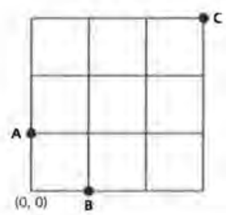
The best location for the warehouse given the locations of the stores is at (2.1, 1.6); this is illustrated in the following graph. The sum of the distances in this case is approximately 5.1 units, the smallest value possible.



7. Suppose the three stores are at points (0, 1), (1, 0), and (3, 3), as shown in the following diagram. Estimate where the location of the warehouse should be. Explain your choice.



8. Using the larger grid provided with this module, plot the location of the three stores as indicated in the previous diagram. Select any three points on the grid to represent the location of the warehouse. Use a piece of string or a computer program to find the distances to each of the stores. Record your results in the following table. Find the sum of the distances.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances (rounded to one decimal place)
____	____	____	____	____
____	____	____	____	____
____	____	____	____	____

9. The best location for the warehouse gives a sum of distances of 4.7 units. Can you find that location?

7. Answers will vary.

8. Answers will vary. One set of possible answers is shown in the chart below.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Sum of the Distances (Rounded to one decimal place)
(1, 1)	1	1	2.8	4.8
(.5, .5)	.7	.7	3.5	4.9
(2, 2)	2.2	2.2	1.4	5.8

9. The best location for the warehouse has coordinates (.8, .8).

10. By trying several locations the best location can be found to be (2.1, 1.2).

11. It should be clear that the best location is somewhere on the interior of the triangle formed by the three points and that the closer the triangle is to being isosceles, the closer the optimum point is to the center of gravity of the triangle. It might not be clear that when we have found the best location for the warehouse and we draw lines between the warehouse and each store, those lines are at 120° angles from one another.

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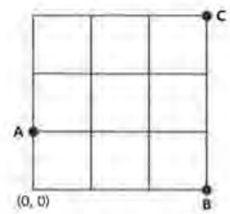
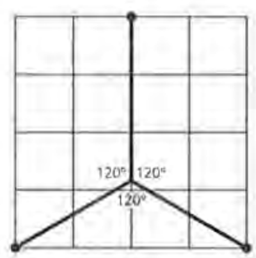
10. Suppose three stores are at points (0, 1), (3, 0), and (3, 3), as shown below. Where should the warehouse be for this arrangement? Explain your estimate.



11. On the basis of the preceding three examples, what can you say about how the best location for the warehouse is related to the locations of the three stores?

You may have noticed that when you have found the best location for the warehouse and you draw lines between the warehouse and each store, those lines are at 120-degree angles from one another.



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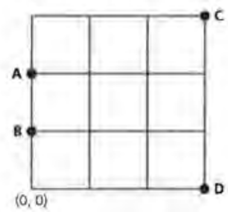
12. Answers will vary.

13. If string is used, then only one decimal place of accuracy is likely. This yields the values in the chart shown below.

Consider Four Points

Suppose there are four stores, rather than three. For example, consider the following configuration of stores A, B, C, and D:



Where should the warehouse be located?

12. What is your initial estimate for the location of a warehouse that minimizes the sum of distances to the four stores? Why did you choose that location?

If the warehouse is at location (1, 1), then the sum of the distances is $\sqrt{2} + 1 + \sqrt{8} + \sqrt{5}$, or about 7.5.

13. Consider locating the warehouse at each of the locations designated in the following chart. Determine the distances required to complete this table by string or computer application. (If estimates are made by way of string, transfer the points to the larger grid provided with this material.)

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Distance to Store D	Sum of the Distances (rounded to one decimal place)
(2, 2)	____	____	____	____	____
(1, 1.5)	____	____	____	____	____
(.5, 1.5)	____	____	____	____	____
(.75, 1.5)	____	____	____	____	____

14. What is the best location for the warehouse?

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Distance to Store D	Sum of the Distances (Rounded to one decimal place)
(2, 2)	2	2.2	1.4	2.2	7.8
(1, 1.5)	1.1	1.1	2.5	2.5	7.2
(.5, 1.5)	.7	.7	2.9	2.9	7.2
(.75, 1.5).	.9	.9	2.7	2.7	7.2

If computer software is used, then more accuracy is possible. Consider the following.

Location of Warehouse	Distance to Store A	Distance to Store B	Distance to Store C	Distance to Store D	Sum of the Distances (Rounded to two decimal places)
(2, 2)	2	2.25	1.41	2.25	7.91
(1, 1.5)	1.13	1.13	2.51	2.51	7.28
(.5, 1.5)	.71	.71	2.92	2.92	7.26
(.75, 1.5)	.90	.90	2.72	2.72	7.24

14. The best location is (.75, .5)

15. The best location is (.25, .5).

16. The best location is (.5, .5).

17. The best location for the warehouse is at the intersection of the line segments formed by connecting each point to the point "opposite" it.

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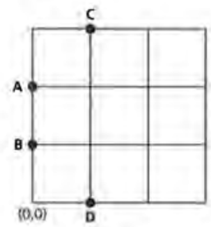
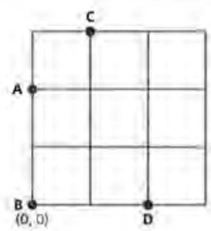
15. Suppose the four stores are at points (0, 2), (0, 1), (1, 3), and (1, 0), as shown below. Now where should the warehouse be?



16. Suppose the four stores are at points (0, 2), (0, 0), (1, 3), and (2, 0), as shown below. Where should the warehouse be?



17. On the basis of the preceding three examples, what can you say about how the best location for the warehouse is related to the locations of the four stores?

You may have noticed that the best location for the warehouse is at the intersection of the line segments formed by connecting each point to the point "opposite" it.

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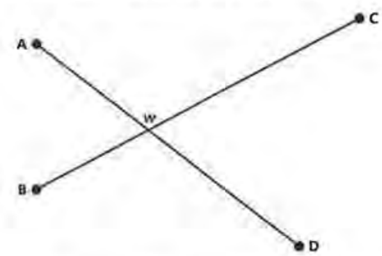
18. The distances from the warehouse to A and to D is not affected, because this sum of distances is just the distance from A to D, which does not change.

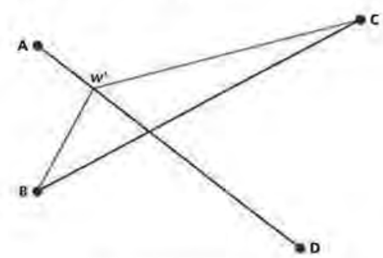
19. The triangle inequality implies that the sum of the distances from the warehouse to B and to C goes up as the location of the warehouse changes from *w* to *w'*. When the warehouse is at location *w*, the sum of the distances is just the distance from B to C. When the warehouse is at location *w'*, the sum of the distances is the sum of the lengths of two sides of the triangle BC*w'*. This is greater than the length of the third side, which is the distance from B to C.

20. When the warehouse is at location *w*, the total distance is the distance from A to D plus the distance from B to C. When the warehouse is at *w'*, the total distance is the distance from A to D plus a value greater than the distance from B to C. Thus, *w* is a better location for the warehouse.

To see that this is the best possible location, consider the following. Suppose the warehouse were located at the point of intersection, labeled as *w* in the following diagram:



Now suppose it is suggested to move the warehouse closer to point A by sliding along the diagonal between A and D. Call this new location *w'*.



18. How does moving the warehouse from location *w* to location *w'* affect the sum of the distances from the warehouse to A and to D? Why?

19. How does moving the warehouse from location *w* to location *w'* affect the sum of the distances from the warehouse to B and to C? Why? (Hint: consider the triangle inequality.)

20. Combine your answers to problems 18 and 19 to show that *w* is a better location than *w'*.

21. From the triangle inequality, we know that the distance from w'' to A and to D is greater than the distance between A and D. Likewise, the distance from w'' to B and to C is greater than the distance between B and C. But if the warehouse is at w, then the total distance is just the distance between A and D plus the distance between B and C. Hence, location w is better than location w''.

If there are five or more points, then there is no simple method to find the best location for the warehouse. Use string and trial and error to find the best location.

Practice and Applications

22. The best location for the warehouse is (.8, 1.8).

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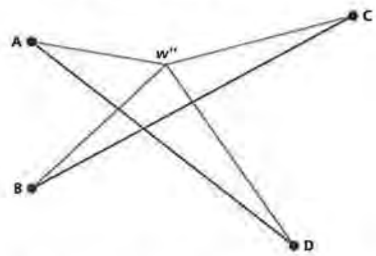
21. Suppose the warehouse were moved to a spot such as w'' in the following diagram, which is not on either diagonal. Explain how the triangle inequality can be applied twice to show that w'' is a worse location than w.



Consider Five or More Points

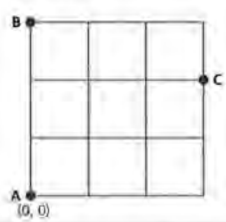
If there are five or more points, then there is no simple method to find the best location for the warehouse. Use string and trial and error to find the best location.

SUMMARY

When there are three stores, the location that minimizes the sum of distances to the stores is given by the spot such that line segments from the stores to the spot meet at 120-degree angles. When there are four points, the location that minimizes the sum of the distances is the intersection of line segments connecting pairs of opposite points. When there are five or more points, then there is no simple method for finding the best location.

Practice and Applications

22. Suppose there are three stores and they are at points (0, 0), (0, 3), and (3, 2), as shown below. Where should the warehouse be?



23. The best location for the warehouse is (1.7, 2.5).

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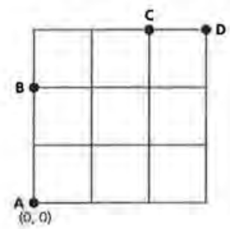
23. Suppose there are four stores and they are at points (0, 0), (0, 2), (2, 3), and (3, 3), as shown on the following diagram. What is the best location of the warehouse?



LESSON 14

The Worst-Case Scenario!

Materials: rulers, *Activity Sheet 13,* Unit V Quiz
Technology: A scientific calculator or graphing calculator will do, but a geometry software package such as Geometer's SketchPad or Cabri is preferable.
Pacing: 2 class sessions

Overview

The Worst-Case Scenario! is an interesting concept that will challenge some students. The problems are designed so that students are presented scattered points on a grid. What center point would minimize the distance needed to travel to the farthest location in this scatter? In other words, where would a center be located to minimize the worst-case scenario? The lesson develops this idea with the students using a "shrinking and sliding" process. Place a constructed circle over the entire scatter of points. As the student shrinks this circle down, it will eventually touch one of the points of the scatter. When this happens, slide the circle so that all of the points are again within the smaller circle. Then start shrinking the circle down again. Repeat this process until it is no longer possible to move the circle without placing points outside the circle. The center of this circle is the estimate of the worst-case scenario. Visualizing this process is again best developed using one of the geometry software packages previously mentioned.

Teaching Notes

The development of the prepared lesson is again designed for the situation in which access to computer software is limited. The specific problems presented in the student materials are arranged to guide the students through the process. Included in this teacher's edition, however, is a demonstration of the "shrink-and-slide" process using the Geometer's Sketchpad. The explanation provides suggestions on how to set up the problems. Specific details related to using the software are not developed for similar reasons mentioned in the opening comments of Lesson 13.

Follow-Up

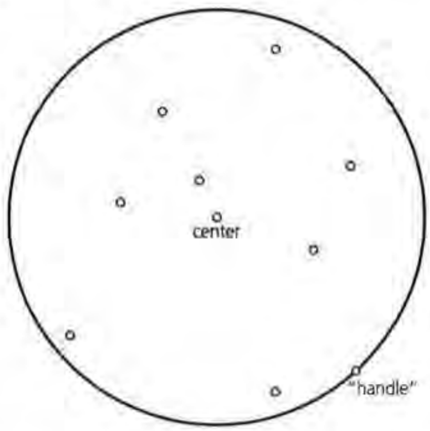
Several applications of "minimizing the worst case" could be presented to students. One discussion that could be particularly relevant is a discussion of insurance rates. The goal of developing insurance rates is to provide for the "worst case" (i.e., car completely wrecked, house completely destroyed). It is also, however, the goal to determine this coverage for a minimum cost to the consumer. Balancing these needs is another example of determining a special type of center! Topics illustrating this concept are found in several business applications.

Technology

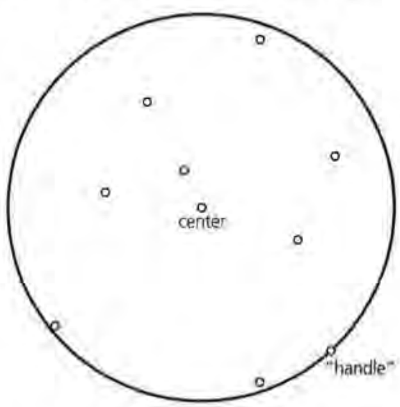
As with Lesson 13, the Geometer's Sketchpad or similar software is helpful when exploring the ideas presented in this lesson. The first part of the lesson (problems 1–9) deals with points on a number line and makes no use of software. However, the second part of the lesson (Problems 10–14) concerns points on the plane. Here software is very useful.

For example, you can use the Geometer's Sketchpad to carry out the "shrink-and-slide" method presented in problem 14 as follows. In the diagram below we have placed eight points on the plane using the

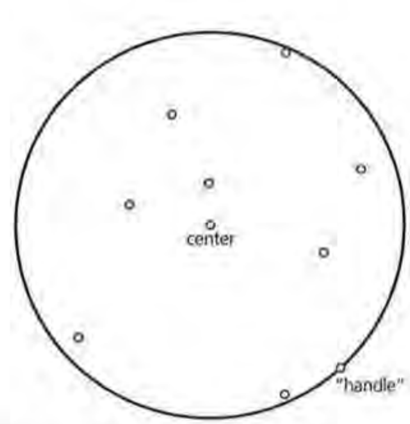
Sketchpad. We wish to find the center point that minimizes the distance to the farthest point. The first step is to choose the circle tool, put the center of the circle at a sensible guess for the solution, and expand the circle until it completely surrounds the eight points. Note that this will add two points to the sketch: the center of the circle and a "handle" that allows you to later shrink the circle.



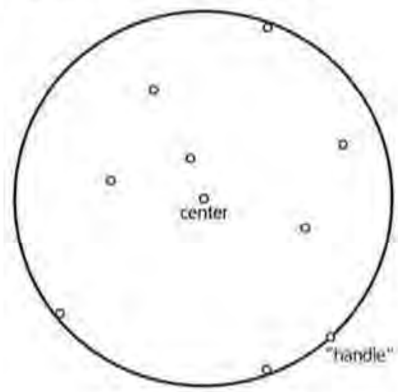
Now grab the handle (use the pointing tool) and shrink the circle until it touches one of the eight points. In the diagram below, the circle touches the point at the lower left part of the sketch.



Now grab the circle by its edge, rather than by the handle, and slide it until there is white space between the circle and each point.

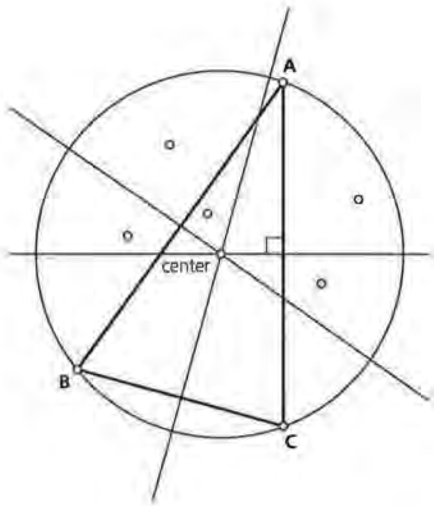


Shrink the circle, as before, until it touches a point. Then slide it; then shrink it; then slide it, etc. Eventually no more sliding or shrinking is possible. When this happens you are finished. The solution to the problem is the center of the circle.



The process of finding the center of a circle by the "shrinking-and-sliding" process approximates the center of a well-known circle involved in geometric constructions. Most scatters of points finally shrink on the three points farthest out. The triangle formed by these three points and the resulting circumscribed circle would locate the point (i.e., circle's center) that minimizes the distances to the farthest points. This circle has a constructed center as the intersection of the perpendicular bisectors. The connection between our shrink-and-slide circle and this circle is exciting; however, this connection is only for acute or right triangles. For obtuse triangles, the shrink-and-slide circle is different.

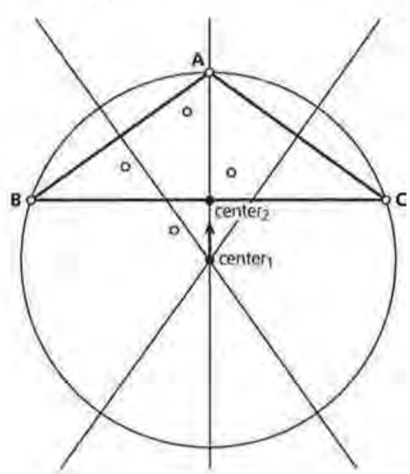
The following graph shows the construction of the circumscribed circle for the example with eight points presented above:



The center of the perpendicular bisectors is labeled above. This is also the center of the smallest circle in our shrink-and-slide process. In this case, the center is the point that minimizes the farthest distance problem.

Obtuse Triangles

It may be that the three points toward the "edge" of the scattering of points form an obtuse triangle, as in the diagram that follows. The center for a circumscribed circle of the three points labeled as A, B, and C is $center_1$. This is not the center that minimizes the distances to the farthest points. $Center_2$ is the center of the circle resulting from our shrink-and-slide discussion. Note its relationship to the points and to the circumscribed circle.



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LESSON 14

The Worst-Case Scenario!

Jacksonville, Florida, is a "port of entry," a U.S. town that receives cars imported from other countries. Would you expect the price of a new car in Jacksonville to be the same as the price of the same model car in Minneapolis?

You have agreed to buy a new car, but the dealer currently does not have the car in the color you want. How does the dealer find and get the car you want?

OBJECTIVE

Calculate the value that minimizes the greatest distance

Reconsider the warehouse problem presented in Lesson 11. Suppose that instead of stocking every store at the end of the week, a different delivery problem is presented. In this new situation, stores call the warehouse when a sale has been made; it is the company's goal to deliver that product from the central warehouse to the store as soon as possible. For example, car dealerships cannot stock every model in every color combination. Once a customer has selected a car with specific features and in a certain color, someone from the central storage warehouse must drive the customer's selection to the dealership. If this is your job, you hope the dealership is close to the "central" warehouse. On some unlucky days, you will be required to travel to the dealership that is farthest from the central warehouse. Thus, the car company considers locating the central warehouse so that *the distance to the farthest dealership is minimized.*

INVESTIGATE

New Car Delivery

This problem and its solution are different from the problem and solution presented in Lesson 11. In Lesson 11, the goal

Solution Key

Discussion and Practice

1. Answers will vary.
2. a. 2

   b. 8

   c. 8, the distance from w to B
3. 5, the distance from w to A or from w to B

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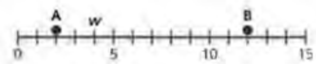
was to minimize the sum of the distances to the stores. The goal in the problems of this lesson is to minimize the *greatest* distance.

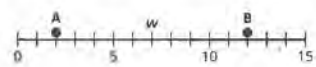
As in previous problems, suppose first that you are concerned with only two stores, A and B, that are at locations 2 and 12 on a number line. A warehouse is to be built between these two stores so that the greatest distance to a store is minimized.

Discussion and Practice

1. What is your initial estimate of the location that minimizes the greatest distance? Why did you choose that location?
2. Suppose the warehouse is located on the number line at $w = 4$. Use the number line that follows to answer the following questions.
   a. What is the distance from w to A?
   b. What is the distance from w to B?
   c. Is the distance greater from the warehouse to store A or to store B?



3. Suppose the warehouse is located at $w = 7$. What is the greater distance from the warehouse to a store?



Which Store is Farther?

Consider walking along the number line. Wherever you are located, you can stop and ask "Which store is farther from here?" If you are to the left of 7, then the answer to this question is "Store B"; if you are to the right of 7, then the answer is "Store A." Putting the warehouse at $w = 7$ is the best we can do; if we move the warehouse away from 7, then either the distance to A gets larger than 5 or the distance to B gets larger than 5.

4. **a.** 1

b. –4.5

c. –2.25

5. Answers will vary.

6. **a.** 2

b. 8

c. 1

d. 8, the distance from *w* to B

7. 5, the distance from *w* to A or from *w* to B

8. Store C is never the store farthest away, since it is not located at the maximum nor the minimum.

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4. For each of the following, determine the location of the point that minimizes the greater distance:

a. Store A is located at –4 and store B is located at 6.

b. Store A is located at –8 and store B is located at –1.

c. Store A is located at –6.5 and store B is located at 2.

Now consider the problem in which there are several stores on a line. Consider stores A, B, and C at locations 2, 12, and 5 on a number line.

5. What is your initial estimate of the location that minimizes the greatest distance? Why did you choose that location?

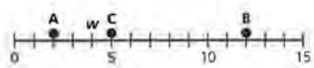
6. Suppose the warehouse is located at $w = 4$.

a. What is the distance from w to A?

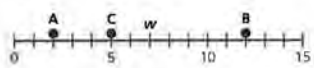
b. What is the distance from w to B?

c. What is the distance from w to C?

d. What is the greatest distance from the warehouse to a store?



7. Suppose the warehouse is located at $w = 7$. What is the greatest distance from the warehouse to a store?



Again, putting the warehouse at $w = 7$ is the best location to answer this question; if the warehouse is moved away from 7, then either the distance to A gets larger than 5 or the distance to B gets larger than 5.

8. Why does the location of C not enter into the consideration of the warehouse's location. (Hint: consider walking along the number line asking yourself, "Which store is farthest away from me?")

9. a. 1.5

b. Store D must be at location 9. The midrange is 4, since that is the location of the dealership, and the minimum is −1. Thus, the maximum must be 9, since (−1 + 9)/2 = 4.

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What happens if there are n stores on a number line? Consider stores $P_1, P_2, \ldots, P_n$ on a number line.

As you walk along this number line, there are only two possible answers to your favorite question, "Which store is farthest from me?" Sometimes store P_1 is the most distant store and sometimes store P_n is the most distant store. If you start to the left of P_1 and walk toward the right, then P_n is the most distant store for the first part of the walk. Eventually P_1 becomes the most distant point. Where does this switch happen?

The ***midrange*** is defined to be:

$$\frac{\text{Minimum value} + \text{Maximum value}}{2}$$

This is another type of center and is the solution to the problem for points located on a number line.

9. Use the midrange to determine the location on a number line of the points requested in the following statements:

   a. Determine the location of point w that minimizes the greatest distance from point w to any one of the following stores: store A at −5, store B at -2, store C at 3, and store D at 8.

   b. The location of the central distribution warehouse for four car dealerships (namely A, B, C, and D) is at the location of 4 on a number line. Store A is located at −1, store B at 2 , and store C at 5. Determine the location of store D. Explain how you determined your solution to this statement.

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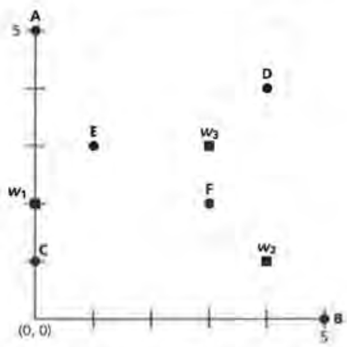
10. Answers will vary.

11. See two tables below.

Now consider a more realistic setting in which the locations of the stores do not fit on a simple number line. A more appropriate representation of the stores is a set of points on a two-dimensional map. Consider the representation of six stores, A–F, as illustrated on an *xy*-coordinate system:



The graph above shows three locations that are being considered for the central warehouse: w_1, w_2, and w_3. The decision of which location to select will be based on the goal of minimizing the greatest distance traveled to any of the stores.

10. Which location do you think should be selected? Why?

11. Collect the information needed to make this decision. Hold a piece of string at location w_1. Measure the length of string to store A and record this length on the summary sheet. Similarly, determine the length of string from w_1 to stores B, C, D, E, and F. Repeat this process by holding the string at w_2 and recording the length to each of the six stores. Also, collect the data using location w_3 as the location of the central warehouse.

Summary Sheet

Coordinate Location of Store (*x*, *y*)	Distance of Store from w_1	Distance of Store from w_2	Distance of Store from w_3
A (0, 5)	3	5.7	3.6
B (5, 0)	5.4	1.4	3.6
C (0, 1)	1	4	3.6
D (4, 4)	4.5	3	1.4
E (1, 3)	1.4	3.6	2
F (3, 2)	3	1.4	1

Summary Sheet

Central Warehouse	Store Farthest from Central Warehouse	Distance from Farthest Store to Central Warehouse
w_1	B	5.4
w_2	A	5.7
w_3	A, B, or C	3.6

12. The location for w_3 is the best of the three suggested locations.

13. Answers will vary. There are locations better than (3, 3).

14. a.–d. Students should follow directions as indicated.

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Summary Sheet

Coordinate Location of Store (x, y)	Distance of Store from w_1	Distance of Store from w_2	Distance of Store from w_3
A	______	______	______
B	______	______	______
C	______	______	______
D	______	______	______
E	______	______	______
F	______	______	______

From the above data, determine the following:

Central Warehouse	Store Farthest from Central Warehouse	Distance from Farthest Store to Central Warehouse
w_1	______	______
w_2	______	______
w_3	______	______

12. From the summary data collected, determine which suggested warehouse location minimizes the distance to the farthest store.

13. What if you were not given the suggested points w_1, w_2, and w_3? Can you find a location that is better than any of these three?

14. Suppose that no restraints are placed on the location of the central warehouse.

a. With a compass, construct a circle that encompasses all six of the points on the map. Construct this circle on a sheet of paper that can be placed over the map points. Your paper should be somewhat transparent so that the map points show through. Remember, it is the center of this circle that matters; therefore, record the center you used to construct this first circle.

b. Using the center of this circle, continue to construct smaller circles that encompass the map points. Repeat this until one of your shrunken circles touches a point on the map.

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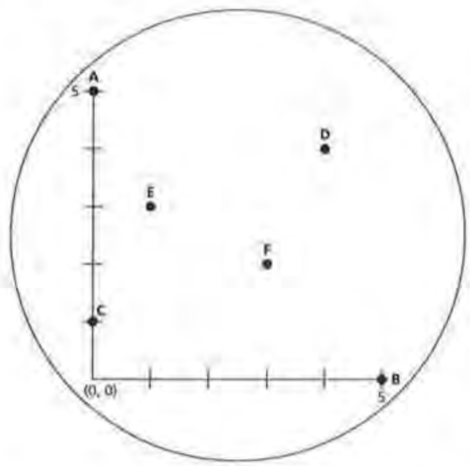
e. The points the circle touches are equidistant from the center of the circle and the circle has been shrunk as much as possible, which makes the greatest distance as small as possible.

f. Stores A and B are farthest from the center of the circle. See lengths in the table below.

c. Now move or **slide** this circle so that all six of the points are enclosed in the circle. Record the new center of the circle you have discovered.

d. Continue experimenting with smaller and smaller circles at this location until the circle touches at least one point. Then slide the circle so that all points are inside this circle.

e. Eventually, you will be no longer able to "**shrink or slide**" the circle and still have it encompass all six points. If you shrink this circle any further, at least one of the points of the map will be located outside of the circle. The **center** of this circle **minimizes the greatest distance.** Why?



f. Complete the chart below. Which store is farthest from the center, *w*, of your circle?

Center w (x, y)	Length to Store A	Length to Store B	Length to Store C	Length to Store D	Length to Store E	Length to Store F
___	___	___	___	___	___	___

Center w (*x, y*)	**Length to Store A**	**Length to Store B**	**Length to Store C**	**Length to Store D**	**Length to Store E**	**Length to Store F**
(2.5, 2.5)	3.5	3.5	2.9	2.1	1.6	.7

Practice and Applications

15. a. 4.5

b. 6

16. The best location is (1.9, 2.5).

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SUMMARY

When dealing with points on a number line, the location that minimizes the greatest distance to any of the points is the ***midrange*** of the values. When dealing with points on a plane, the location that minimizes the greatest distance to any of the points is the center of the smallest circle that encompasses all of the points.

Practice and Applications

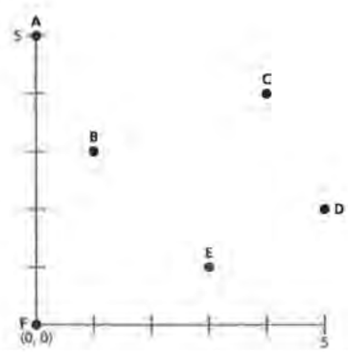
15. For each of the following, develop a number line and plot the specific points. Use the midrange to determine the location on your number line of the points requested in the following statements:

a. Determine the location of the point that minimizes the greatest distance to any one of the following stores: store A at 2, store B at 5, and store C at 7.

b. Determine the location of the point that minimizes the greatest distance to any one of the following stores: store A at –3, store B at –1, store C at 2, store D at 11, and store E at 15.

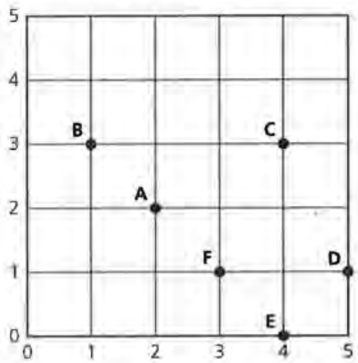
Consider the location of stores for each of the situations plotted on the following *xy*-coordinate systems. Using the "shrink and slide" ideas, approximate the center point that minimizes the location from the center to the point farthest from the center. If possible, develop these problems using a geometric software program like Sketchpad or Cabri.

16.



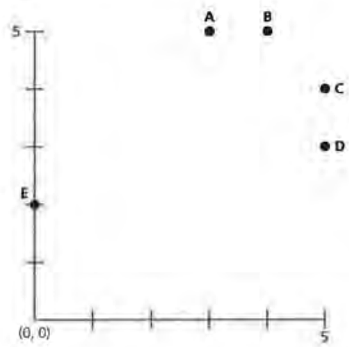
17. The best location is (2.6, 2.8).

18. Answers will vary. The key step in placing five stores on the map to satisfy the condition that the center be at (3, 2) is to construct a circle that has (3, 2) as its center. Here is one example.



STUDENT PAGE 136

17.



18. For this problem, create a map similar to the type presented in problems 16 and 17. Design your map for at least five stores that have a *center* at the location of (3, 2) on an *xy*-coordinate system.

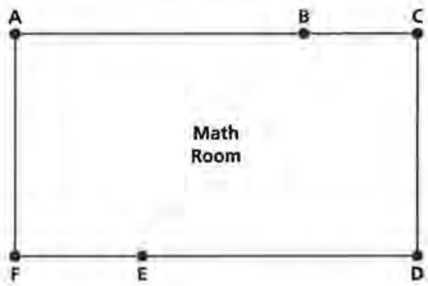
Teacher Resources

UNIT I QUIZ

Estimating Centers of Measurements

NAME ____________________

The following is a rough sketch of the rectangular math room at Rob's high school.



The labels indicated in this sketch were marked on the floor and used by three groups of students to record the following measurements in meters.

	AB	BC	CD	DE	EF	FA
Group 1	15.3	5.8	12.7	11.3	3.1	10.8
Group 2	15.5	5.8	12.2	18.4	3.2	13.1
Group 3	15.6	6.0	12.3	18.6	3.0	12.9

1. Rob used these measurements to develop a scale drawing of the math room:

	AB	BC	CD	DE	EF	FA
Rob's estimates	15.5	5.8	12.4	16.1	3.1	13.0

Describe how you think Rob selected his estimates for each of these lengths.

a. AB **b.** BC **c.** CD

d. DE **e.** EF **f.** FA

2. What properties of a rectangle could be used to evaluate the accuracy of Rob's estimates?

3. How do you know a scale drawing of the math room using Rob's estimates will be inaccurate?

4. Develop a scale drawing of the math room using Rob's estimates and the scale 1 cm = 1 meter.

5. Identify one of Rob's estimates you think is an inaccurate estimate of the data set.

   a. Describe why you selected this measurement.

   b. What value would you use for this measurement? Why?

6. Re-examine the measurements recorded by the three groups. Select an estimate for each of the labeled distances. Copy and complete the following chart indicating your estimates.

	AB	**BC**	**CD**	**DE**	**EF**	**FA**
Your estimates						

7. Develop a scale drawing of the math room using your estimates and the scale of 1 cm = 1 meter.

8. Do you think your drawing is a more accurate representation of the math room than Rob's drawing? Why or why not?

9. Future plans indicate this room will be carpeted. Estimate the cost of carpeting this room if a contract for carpet is arranged at $14.75 per square meter. Identify at least one factor that would explain why your estimate might not be accurate.

10. Jenny walked a distance of 30 yards several times. Each time she counted the number of steps it took her to complete this distance. The following data represents the results Jenny recorded for five of her walks.

38, 40, 42, 40, 41

Using the data set collected by Jenny, determine Jenny's

a. mean number of steps

b. median number of steps

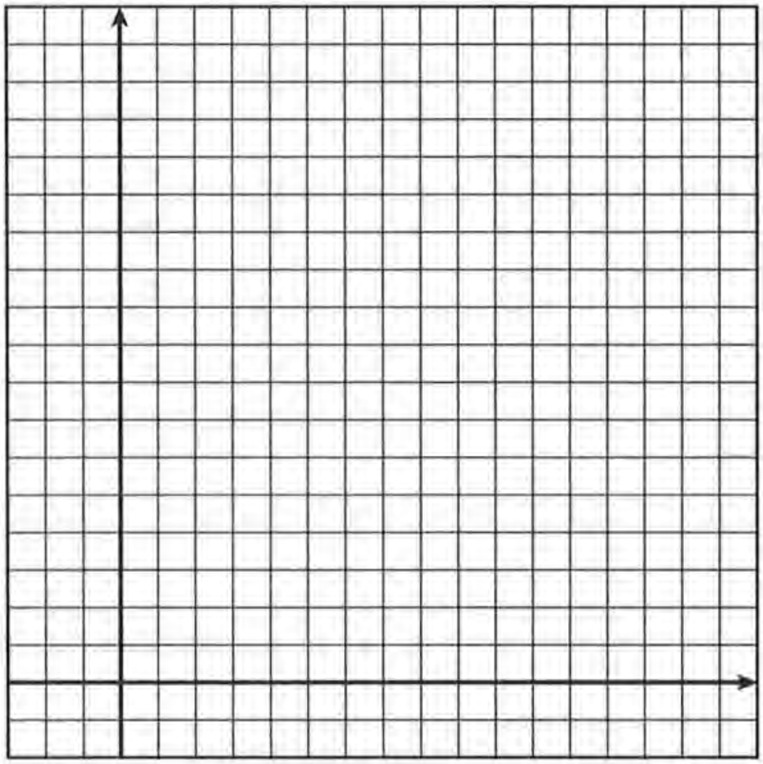
c. mode number of steps (if any)

11. Jenny collected the mean number of steps it took several other members of her school to walk the 30-yard distance. Jenny also determined the leg length in inches for each person in her experiment. She recorded this additional piece of data in the following chart.

Name	Number of steps (x)	Length of leg in inches (y)
Anthony Smith	38	42
Matthew Denizen	40	40
Nicole Willis	42	38
Randal Feingold	40	41
James Brooks	41	39
Anthony Balistreri	43	36
Rachel Jones	44	35
Jeffrey Scott	40	40
Claud Salyards	42	67

Study this data set. Jenny might have erred in measuring! Jenny most likely erred in measuring leg length of which person? Explain why you think her measurement was in error.

12. Develop a **scatter plot** of the data given in problem 11. Use a coordinate grid similar to the one below. *Do not* include any values from the data set you think represent an error in Jenny's measurement.



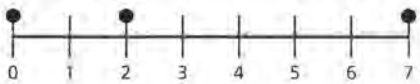
13. Do you think you could estimate a person's leg length based on the number of steps it took this person to walk 30 yards? Why or why not?

14. Estimate Jenny's leg length. How did you determine your estimate?

Centers of Balance

NAME ______________________________

1. Think of the number line below as a ruler with raisins on it.



a. Develop a sketch of this arrangement if the broad side of a pencil is placed crosswise under the ruler at location 5.

b. Develop a sketch of this arrangement if the broad side of a pencil is placed crosswise under the ruler at location 1.

c. At what location would the broad side of a pencil balance this arrangement?

2. The following table represents the arrangement of objects of unequal weight along a number line.

Point	Units of weight W_i	Position on the number line x_i	Weighted value $W_i x_i$
A	5	30	
B	2	50	
C	3	70	
D	1	90	

a. Draw a number line and mark off equal segments to represent the appropriate values from this table.

b. Continue to develop your sketch by using the symbol • as directed by the data in the table.

c. Determine the weighted mean of your picture.

d. "A recent quiz in a geometry class indicated the following results: five students received a score of 30%, two students received a score of 50%, . . ." Complete the description of this problem if the previous chart summarized the results for this particular geometry quiz. Indicate what question is asked by your problem.

e. What was the average score (in percent) of the geometry quiz discussed in part d?

3. Mr. Witmer lost Maggie's chemistry quiz. His class of seven students averaged 82% on this exam. Four students received a score of 75% and two students received a score of 90%.

a. Sketch a number line of the information given for the six students. Use the symbol • to represent each student's score on the number line.

b. Set up and solve for Maggie's score.

4. Ten high school students responded to this statement: "Estimate your income from part-time jobs during this past school year." The following chart organizes the responses received. Determine the mean of the estimated incomes for the 10 students.

Estimated income	Number of students
\$0	2
\$700	3
\$1500	3
\$2000	1
\$2400	1

UNIT III QUIZ

"Raisin" Country

NAME ____________________

1. Using a straightedge, develop a sketch for each of the following:

   a. an equilateral triangle

   b. an obtuse triangle

   c. a concave quadrilateral

   d. a convex quadrilateral

   e. a pentagon with a centroid "outside" it

2. One raisin was taped at each of the vertices of the isosceles triangle XYZ. The balance point was determined and labeled as location B.

Sketch triangle XYZ and point B. In addition, construct each of the following segments.

   a. Extend $\overline{XB}$ until it intersects $\overline{ZY}$ at P_1.

   b. Extend $\overline{YB}$ until it intersects $\overline{XZ}$ at P_2.

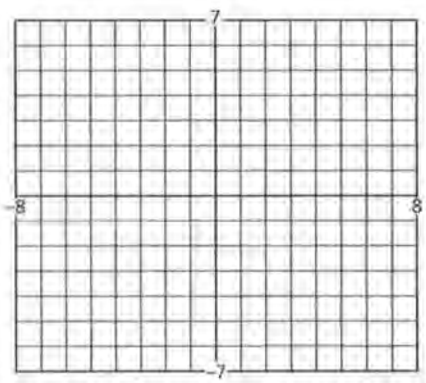
   c. Extend $\overline{ZB}$ until it intersects $\overline{XY}$ at P_3.

   d. How many pairs of congruent triangles are formed by the figure you sketched? Name each pair of congruent triangles you identified.

3. Quadrilateral $P_1P_2P_3P_4$ was cut out of poster paper and placed on an xy-coordinate system. The following data were recorded regarding the placement of the quadrilateral:

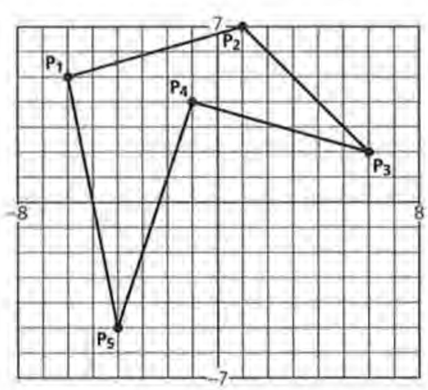
Points	
P_1	(7, 6)
P_2	(3, −6)
P_3	(−5, −7)
P_4	(−6, 5)

a. On an xy-coordinate graph, locate the vertices and draw this quadrilateral.



b. If a raisin were taped at each of the vertices (and each raisin were of equal weight), determine the location of the balance point, or centroid. Explain your method.

4. A pentagon $P_1P_2P_3P_4P_5$ was cut out of poster paper and placed in an xy-coordinate graph. This pentagon was arranged as indicated in the graph below:



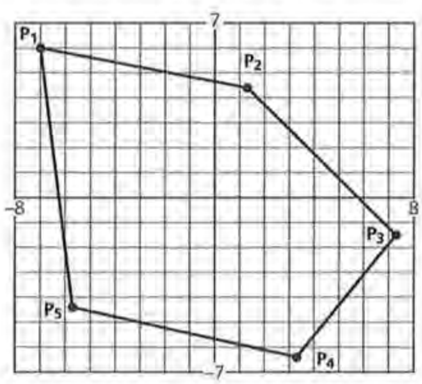
a. Consider a raisin taped at each of the vertices P_1, P_2, P_3, P_4, and P_5. Given the placement of the pentagon in the graph, copy and complete the chart that follows.

Points P_i	Number of raisins	x-coordinate	y-coordinate
P_1	1		
P_2	1		
P_3	1		
P_4	1		
P_5	1		

b. Determine the *xy*-location of the balance point for this pentagon.

c. Would you be able to determine the balance point of this pentagon by balancing the cut-out figure at the end of a pencil or similar fulcrum? Explain your answer.

5. A pentagon similar to the one presented in the two lessons of this section is shown below on a coordinate grid.



a. A total of six raisins are to be taped to the vertices of this pentagon. At least one raisin must be taped to each vertex. How many different ways could six raisins be arranged to outline this pentagon? Identify the number of raisins at each vertex for each arrangement.

b. Estimate the location of balance point B_1 based on one of the arrangements of six raisins from part a. Explain the method you used to determine your estimate.

c. *Challenge problem:* A total of seven raisins are to be taped to the vertices of this pentagon. How many different balance points could be calculated to balance the seven raisins outlining the pentagon? (Remember, at least one raisin must be taped at each vertex.)

Population Centers

NAME ______________________________

Futurists speculate on the population distribution of our country in the next 50 to 100 years. Although estimates this far in the future are primarily speculative, estimates of population trends within the next 10 to 20 years are considered very important in planning and governing our country. The population of the United States has grown approximately 9% every 10 years. During the 1980 to 1990 decade, a 9% growth represented an additional 20 million people added to our country's population.

Lesson 10 indicated that an approximate location of the population center of the United States can be determined by working with the most populous states. Consider the 12 most populous states from the 1990 Census:

State (abbreviation)	**Population** P_i (in thousands)
CA	29,760
NY	17,990
TX	16,987
FL	12,938
PA	11,882
IL	11,431
OH	10,847
MI	9,295
NJ	7,730
NC	6,629
GA	6,478
VA	6,187

1. Determine the total 1990 population of the 12 states.

2. A population increase of 9% is predicted for the nation for the ten-year period from 1990 to 2000. If so, what will be the approximate increase in population in the 12 states by the year 2000?

3. Assume none of the 12 states will lose population during the 1990s.

a. Distribute the number of people you estimated in problem 2 so that the country's population center for the year 2000 moves **northeast** of the country's 1990 population center as indicated by the Bureau of the Census. Record your figures for each state's population in 2000 in such a way as to cause this shift in the population center.

State	Population 1990 (in thousands)	Population 2000 (in thousands)
CA	29,760	
NY	17,990	
TX	16,987	
FL	12,938	
PA	11,882	
IL	11,431	
OH	10,847	
MI	9,295	
NJ	7,730	
NC	6,629	
GA	6,478	
VA	6,187	

b. Use your map of the United States from Lesson 10 that includes the *xy*-coordinates (*Activity Sheet 9*) to help you determine the population center for the year 2000. Start by copying and completing the table on the next page. You will need to use the 2000 population estimated for each of the 12 states in part a.

State	Population 2000 P_i (in thousands)	x-coordinate	y-coordinate	$P_i x_i$	$P_i y_i$
CA					
NY					
TX					
FL					
PA					
IL					
OH					
MI					
NJ					
NC					
GA					
VA					

Calculate the centroid for the estimated population center in 2000. Did the population center shift to the northeast as expected?

c. This time, distribute the estimated population increase among the 12 states so that the country's population center for the year 2000 moves **southwest** of the country's 1990 population center. Record your estimate of each state's population in 2000 in such a way as to cause this shift in the population center.

State	Population 1990 (in thousands)	Population 2000 (in thousands)
CA	29,760	
NY	17,990	
TX	16,987	
FL	12,938	
PA	11,882	
IL	11,431	
OH	10,847	
MI	9,295	
NJ	7,730	
NC	6,629	
GA	6,478	
VA	6,187	

d. Copy and complete the following table to help you determine your estimate for a population shift to the southwest.

State	Population 2000 P_i (in thousands)	x-coordinate	y-coordinate	P_ix_i	P_iy_i
CA					
NY					
TX					
FL					
PA					
IL					
OH					
MI					
NJ					
NC					
GA					
VA					

Calculate the centroid for your estimated population center in 2000. Did the population center shift to the southwest?

4. Consider your results from problem 3.

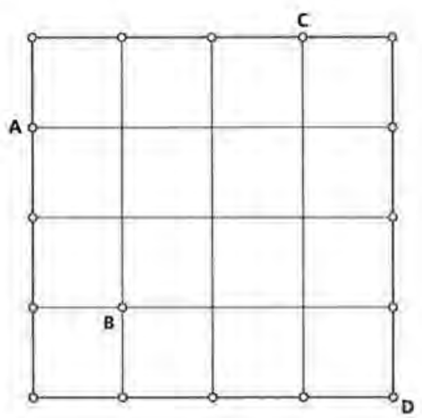
a. What conditions might cause a population shift to the northeast?

b. What conditions might cause a population shift to the southwest?

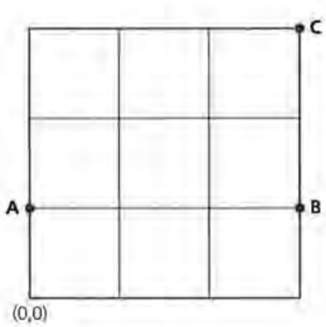
Minimizing Distances by a Center

NAME ______________________________

1. Copy the grid below and consider the four points A, B, C, and D. Locate the center that minimizes the sum of the distances by using helicopter geometry. Label this center as C_1.



2. Copy the grid below and consider the three points A, B, and C.



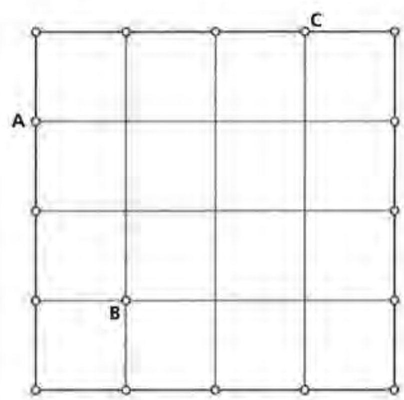
a. Locate the center that minimizes the sum of the distances using taxicab geometry. Label this center as C_1.

b. Locate the center that minimizes the sum of the distances using helicopter geometry. Label this center as C_2.

c. Locate the center that minimizes the greatest distance to any of the points. Label this center as C_3.

3. Consider taping a raisin to each of the points A, B, and C in the diagram from problem 2. Determine the center of gravity of the three raisins. Label this point as C_4.

4. Copy the grid below and consider the three points A, B, and C.



a. Locate the center that minimizes the greatest distance to any of the points. Label this center as C_1.

b. Add a fourth point, D, to the diagram such that the center that minimizes the greatest distance to any of the points does not change.

c. Add a fifth point, E, to the diagram such that the center that minimizes the greatest distance to any of the points *does* change.

Estimating Centers of Measurements

1. **a.** AB; median

   b. BC; median or mode

   c. CD; mean

   d. DE; mean

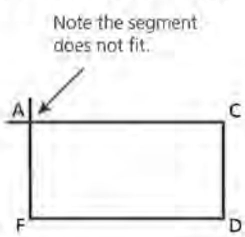
   e. EF; median

   f. FA; Rob threw away the recorded value of 10.8 as it is highly suspect to error. Then he determined the mean of the remaining two values.

2. The widths should be equal if the figure is a rectangle; therefore, AF = CD. Similarly, the lengths should be equal, or AC = FD.

3. As each measurement is an estimate, the accumulated effect of the measurements will show up when the last segment does not complete a rectangle.

4. A rough sketch (not drawn to the scale expected of the students) would look similar to the following:



5. The segment most suspect to error is DE.

   a. It was based on the mean of three measures that appear to have one measure highly suspect to an error. This error contributes to a summary of DE that makes it also suspect to error.

   b. DE = 18.5 m

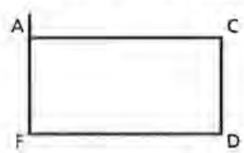
   This value is determined by throwing out the 11.3 measure (probably a recording error) and averaging the remaining values.

6. The following values correct the DE value as indicated:

	AB	BC	CD	DE	EF	FA
Your estimate	15.5	5.8	12.4	18.5	3.1	13.0

(Note: The difference in CD and FA could also be summarized by students. As these two measures should be equal, a discussion of possibly using the median or similar summary might be considered.)

7. A rough sketch would indicate a slightly better fit to the rectangle.



8. The accumulated errors of the set of values in problem 6 more closely form a rectangle.

9. If the measures 13.0 m for width and 21.6 m for length are used, then the area is

   $(13.0\text{ m})(21.6\text{ m}) = 280.8\text{ m}^2$
   or approximately 281 m^2.

   The cost of carpeting this room would be estimated as follows:

   $(281\text{ m}^2)(\$14.75) = \$4144.75$,
   or approximately \$4145

The estimate may be inaccurate due to

i. errors in recording the measurements;

ii. differences in the measuring instruments;

iii. placement of a meter stick end to end to measure the distances.

10. The data set in order is

38, 40, 40, 41, 42.

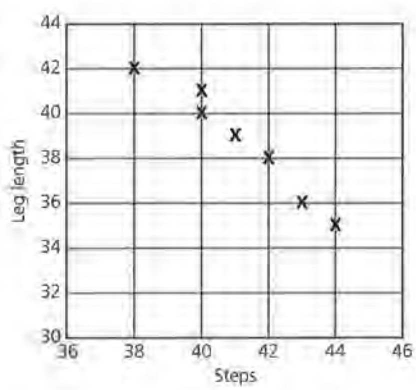
a. The **mean** number of steps is 40.2.

b. The **median** number of steps is 40.

c. The **mode** number of steps is 40.

11. Claud Salyards' leg length of 67 inches is highly suspect to error. Apparently it is simply an inaccurate recording of his measure.

12. The following is a scatter plot of the data (not including Claud's point).

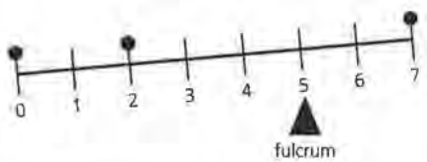


13. The values appear to be connected. A linear relationship is suggested by the scatter plot.

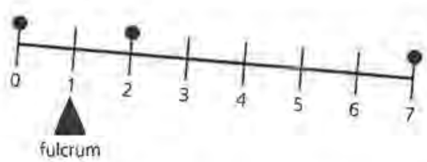
14. Using 40 steps as Jenny's description, the line suggested by the graph would place her leg length as 40 or 41 inches.

Centers of Balance

1. a. One possible sketch of this arrangement with the broad side of a pencil placed underneath the ruler at location 5:



b. One possible sketch of this arrangement with the broad side of a pencil placed underneath the ruler at location 1:

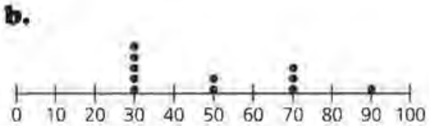


c. A pencil at location 3 would balance the ruler.

$$\bar{x} = \frac{0+2+7}{3} = \frac{9}{3} = 3$$

2. a. See the completed development of the number line in part b.

b.

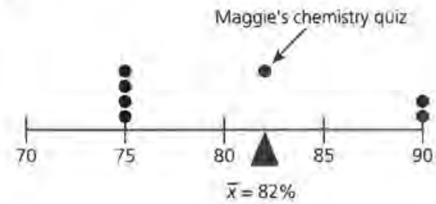


c. $\bar{x} = \frac{5(30)+2(50)+3(70)+1(90)}{11} = \frac{550}{11} = 50$

d. "A recent quiz in a geometry class indicated the following results: five students received a score of 30%, two students received a score of 50%, three students received a score of 70%, and 1 student scored 90%. What was the average score (in percent) of this quiz?"

e. The average score was 50%. (Note: It is important to indicate that this mean is a percent.)

3. a.



b. $\bar{x} = 82\% = \frac{4(75\%)+2(90\%)+1(x)}{7}$

$7(82) = 4(75) + 2(90) + 1x$

$574 = 300 + 180 + x$

$574 = 480 + x$

$x = 94$

Maggie's score was 94%.

4. $\bar{x} = \frac{2(\$0)+3(\$700)+3(\$1500)+1(\$2000)+1(\$2400)}{10}$

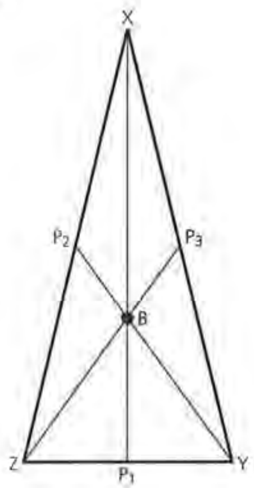
$\bar{x} = \frac{0+\$2100+\$4500+\$2000+\$2400}{10}$

$\bar{x} = \frac{\$1100}{10} = \$1100$

"Raisin" Country

1. a.–e. Answers will vary. The goal of this problem is for students to demonstrate an understanding of the definitions of key geometric figures. Information to complete this problem was not introduced in the lessons but should have been researched (or previously introduced).

2.



a. Follows directions—see above diagram.

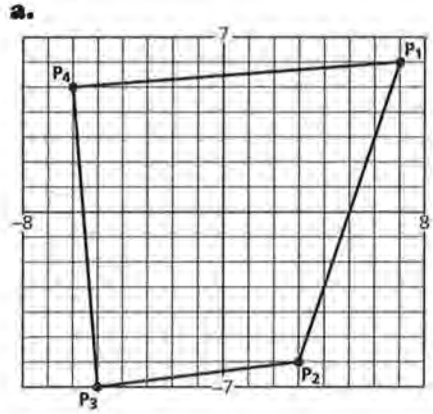
b. Follows directions—see above diagram.

c. Follows directions—see above diagram.

d. 7 pairs

ΔXBP_2 congruent to ΔXBP_3
ΔXYP_2 congruent to ΔXZP_3
ΔXBZ congruent to ΔXBY
ΔP_2BZ congruent to ΔP_3BY
ΔXP_1Z congruent to ΔXP_1Y
ΔBZP_1 congruent to ΔBYP_1
ΔP_2YZ congruent to ΔP_3ZY

3. a.



b. The centroid is C(–0.25, –0.5)

The method used to find this centroid involves the means of the x-coordinates and the y-coordinates.

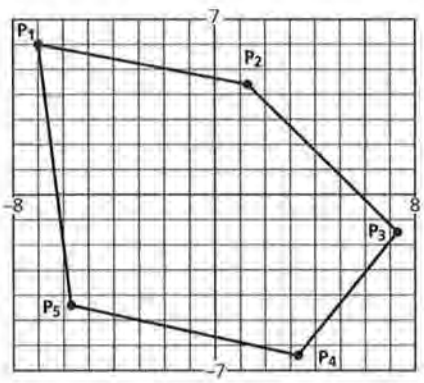
4. a.

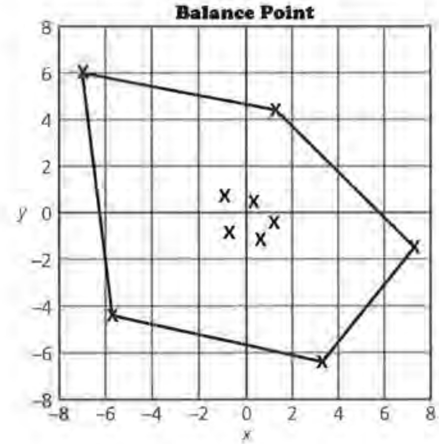
Points P_i	Number of raisins	x-coordinate	y-coordinate
P_1	1	–6	5
P_2	1	1	7
P_3	1	6	2
P_4	1	–1	4
P_5	1	–4	–5

b. The balance point for the above (by way of the means) is C (–.8, 2.6).

c. As this shape is concave, there is a likelihood the balance point lies outside of the shape. The above balance point indicates this is the case. Therefore, any attempts to balance this shape with poster paper or raisins would not be possible.

5.





a. By taping the additional raisin to a vertex (producing one vertex with a weight of 2), there are five arrangements that could be developed with a total of six raisins. A centroid for each arrangement is indicated in the table below. Students are expected to determine the centroid for only one of the arrangements.

b. Look at the table at the end of column 2. Values for this table were derived using a spreadsheet. The balance point for each example is plotted on the graph developed from a charting option of the spreadsheet used to calculate the balance points. This visual is particularly interesting because it shows the shift of the balance point to the heavier vertex of each example.

c. *Challenge problem*

(Problem is intended to get students to think of patterns.)

There are five ways in which an additional 2 raisins can be taped at each vertex for a total of 7 raisins. Now for the harder part. If 1 additional raisin is taped at P_1, there are 4 ways the remaining raisin can be taped to the other vertices. Likewise, if one raisin is taped to P_2, then there are only three ways (as we already counted one in the previous arrangement) to place the other raisins at different vertices. P_3 would have 2, P_4 would have 1, and then: $4 + 3 + 2 + 1 = 10$. Therefore, there are 5 + 10, or 15 ways to arrange the seven raisins.

(5a)

	Raisins located at P_1	Raisins located at P_2	Raisins located at P_3	Raisins located at P_4	Raisins located at P_5	Balance point B_1
Example 1	2	1	1	1	1	(−1.2, 0.7)
Example 2	1	2	1	1	1	(0.25, 0.4)
Example 3	1	1	2	1	1	(1.25, −0.6)
Example 4	1	1	1	2	1	(0.6, −1.4)
Example 5	1	1	1	1	2	(−0.9, −1.1)

(5b)

State	Population P_i (in thousands)
CA	29,760
NY	17,990
TX	16,987
FL	12,938
PA	11,882
IL	11,431
OH	10,847
MI	9,295
NJ	7,730
NC	6,629
GA	6,478
VA	6,187

Population Centers

1. The sum of the populations for the twelve states is 148,154 thousand people, or 148,154,000 people. (This represents approximately 60% of the people in the country in 1990.)

2. 9% of the number of people in the twelve states is:

(148,154,000)(.09) = 13,334,000 (approximate value).

3. a. Distribute as many of the added people to the states in the northeast. One example of this type of distribution is included in the chart at the right. Asterisks indicate states increasing their populations.

State	Population 1990 in thousands)	Population 2000 (in thousands)
CA	29,760	29,760
*NY	17,990	23,990
TX	16,987	16,987
FL	12,938	12,938
*PA	11,882	17,882
IL	11,431	12,431
*OH	10,847	11,181
MI	9,295	9,295
NJ	7,730	7,730
NC	6,629	6,629
GA	6,478	6,478
VA	6,187	6,187

b. The centroid for the data set below is (0.33, 1.9). Comparing this new centroid to C (–0.5, 1.7), the approximate location of the 1990 population center, we see a shift in the population to the northeast. (See note at the end of this quiz.)

State	Population 2000 P_i (in thousands)	x-coordinate	y-coordinate	$P_i x_i$	$P_i y_i$
CA	29,760	–11.5	2.5	–342,240	74,400
*NY	23,990	5.5	4.5	131,945	107,955
TX	16,987	–3.0	–2.5	–59,454.5	–42,467.5
FL	12,938	2.5	–2.0	45,283	–25,876
*PA	17,882	5.0	3.0	89,410	62,587
IL	12,431	0.5	2.0	6,215.5	31,077.5
*OH	11,181	2.5	2.5	33,543	27,952.5
MI	9,295	2.0	3.5	18,590	37,180
NJ	7,730	5.5	3.25	46,380	25,122.5
NC	6,629	4.5	1.0	33,145	6,629
GA	6,478	2.5	–0.5	19,434	–3,239
VA	6,187	5.0	1.5	30,935	12,374

c. The 13,334,000 added people to the population is focused in the southwestern states. 8000 (thousand) people were added to California, 5000 (thousand) to Texas, and 334 (thousand) people to Florida. Many other distributions could be developed by the students.

State	Population 1990 in thousands)	Population 2000 (in thousands)
*CA	29,760	37,760
NY	17,990	17,990
*TX	16,987	21,978
*FL	12,938	13,272
PA	11,882	11,882
IL	11,431	11,431
OH	10,847	10,847
MI	9,295	9,295
NJ	7,730	7,730
NC	6,629	6,629
GA	6,478	6,478
VA	6,187	6,187

d. The centroid for the data set below is (–0.7, 1.7). Comparing this new centroid to C (–0.5, 1.7), the approximate location of the 1990 population center, we see a slight shift in the population to the southwest.

State	Population 2000 P_i (in thousands)	x-coordinate	y-coordinate	$P_i x_i$	$P_i y_i$
*CA	37,760	–11.5	2.5	–434,240	94,400
NY	17,990	5.5	4.5	98,945	80,955
*TX	21,978	–3.0	–2.5	–76,923	–54,945
*FL	13,272	2.5	–2.0	46,452	–26,544
PA	11,882	5.0	3.0	59,410	41,587
IL	11,431	0.5	2.0	5,715.5	28,577.5
OH	10,847	2.5	2.5	32,541	27,117.5
MI	9,295	2.0	3.5	18,590	37,180
NJ	7,730	5.5	3.25	46,380	25,122.5
NC	6,629	4.5	1.0	33,145	6,629
GA	6,478	2.5	–0.5	19,434	–3,239
VA	6,187	5.0	1.5	30,935	12,374

4. a. Conditions of climate and employment might cause a shift in the center of the country to that direction. This shift would be a result of reversing some of the conditions currently noted by the Census Bureau. These conditions might possibly include: a climate change that makes the cooler areas more attractive; issues related to availability of water; and issues related to improved conditions for the industries of this area.

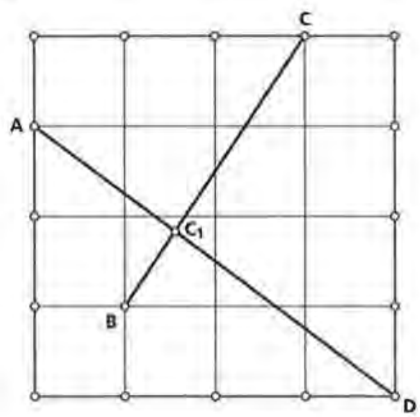
b. If current conditions continue, then the shift to the southwest will continue. Issues related to people wanting warmer climates, expansion of the migration of citizens from Mexico, and similar trends would cause this expansion to the southwest.

One of the points of this problem is for students to realize that a shift in the country's population center for these two examples requires the growth of different states and different regions of the country. This is how the population center becomes an indicator of more than just a location.

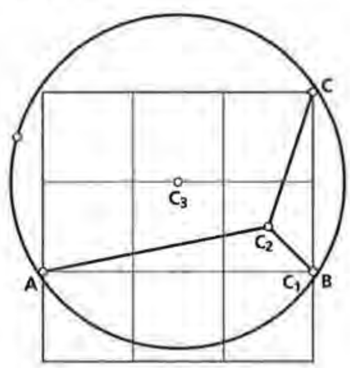
Note: The U.S. map provided on *Activity Sheet 9* could be used by the students to decrease time needed to complete **"The Big Picture"** component of this quiz. The U.S. map has locations of population centers based on state capitals for the 48 connected states. The coordinates used in the previous problems were obtained from this map. Students using this map would not need to superimpose a coordinate system over a U.S. map; in addition, students would not need to determine locations of capitals. If necessary, the coordinate values recorded in *Activity Sheet 11* of this teacher's edition could also be given to the students as a handout or as a spreadsheet. This again would allow flexibility in developing a timetable for completing this lesson.

Minimizing Distances by a Center

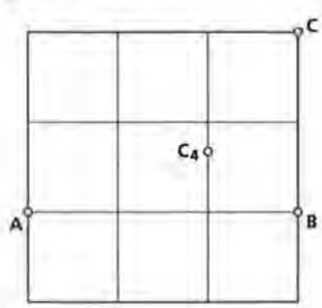
1. The center C_1 is shown on the following diagram.



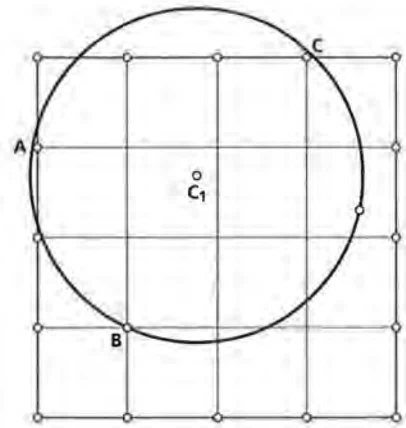
2. The centers C_1 (which is the same as point B), C_2, and C_3 are shown in the following diagram.



3. The center C_4 is shown on the following diagram.



4. a. The center that minimizes the greatest distance to any of the points is labeled as C_1.



b. Any point, D, inside the circle will keep the center that minimizes the greatest distance to any of the points the same.

c. Any point, E, outside the circle will cause the center that minimizes the greatest distance to any of the points to change.

Data Summary 1
Lesson 1: Problem 15a

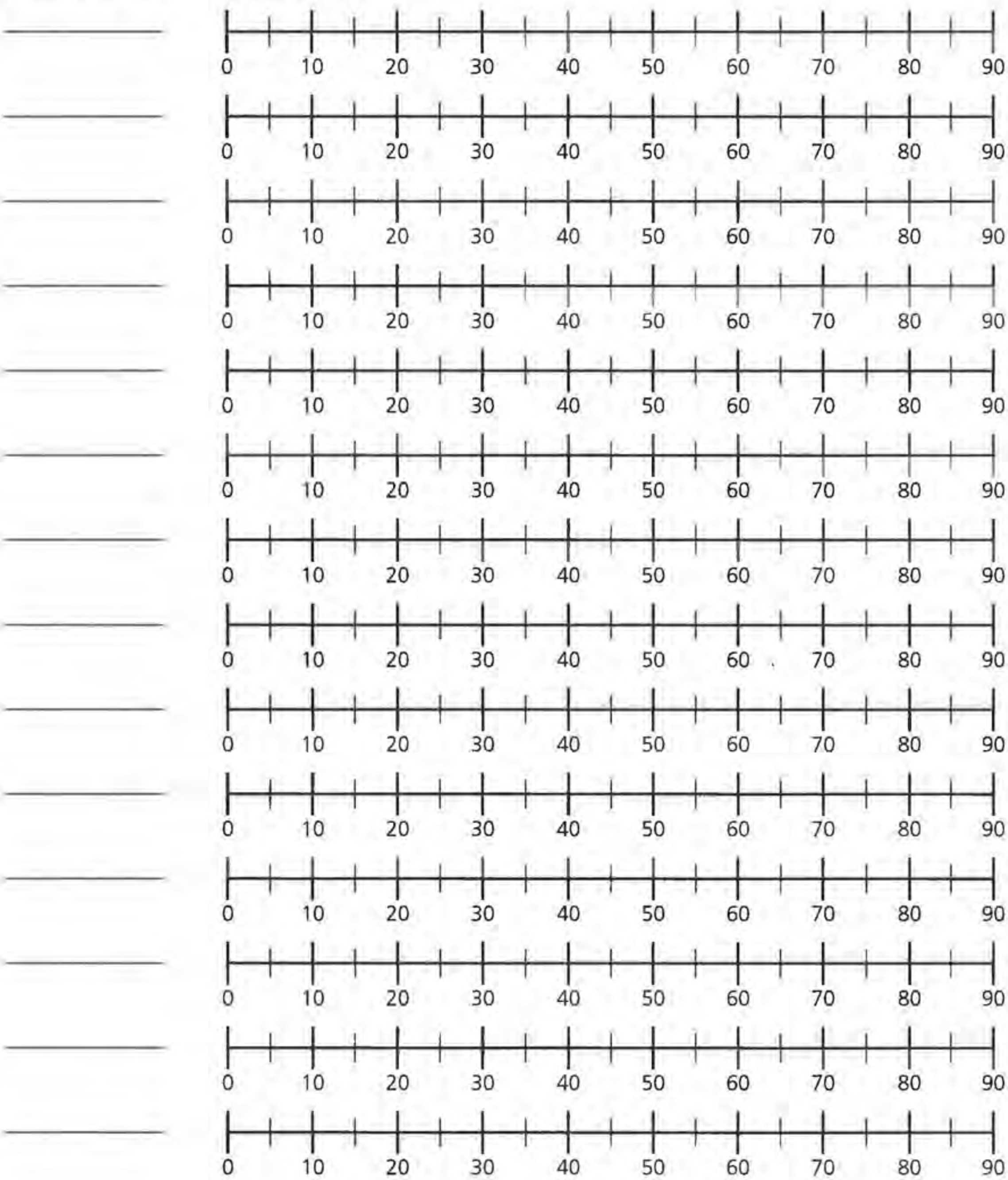
NAME ____

Measurement of segment to be used in your sketch	Criteria used for this measurement	Why?
AB =		
BC =		
CD =		
DE =		
EF =		
FG =		
GH =		
HI =		
IJ =		
JK =		
KL =		
LM =		
MA =		

Data Summary 2
Lesson 2: Problem 16

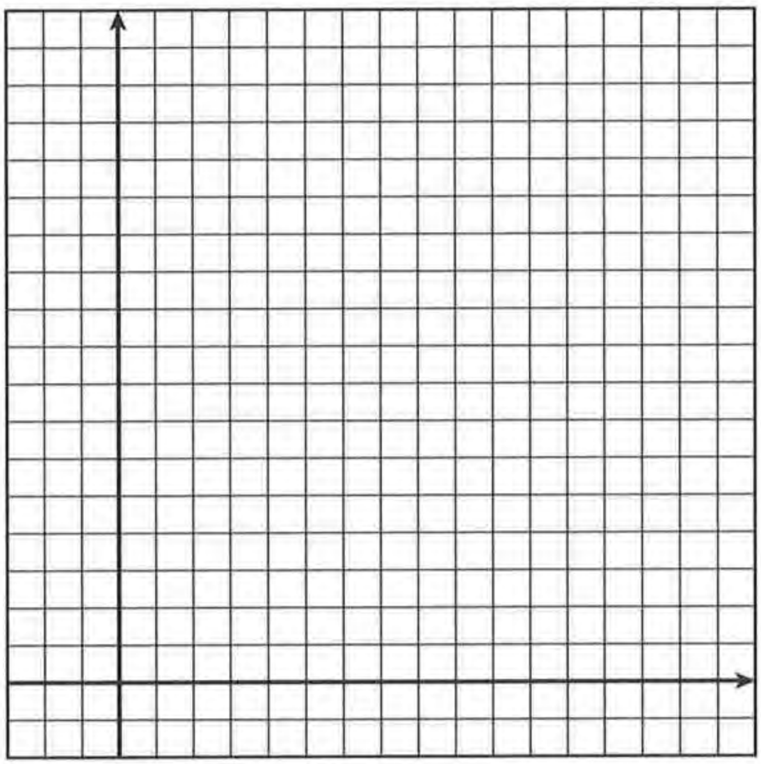
NAME ____

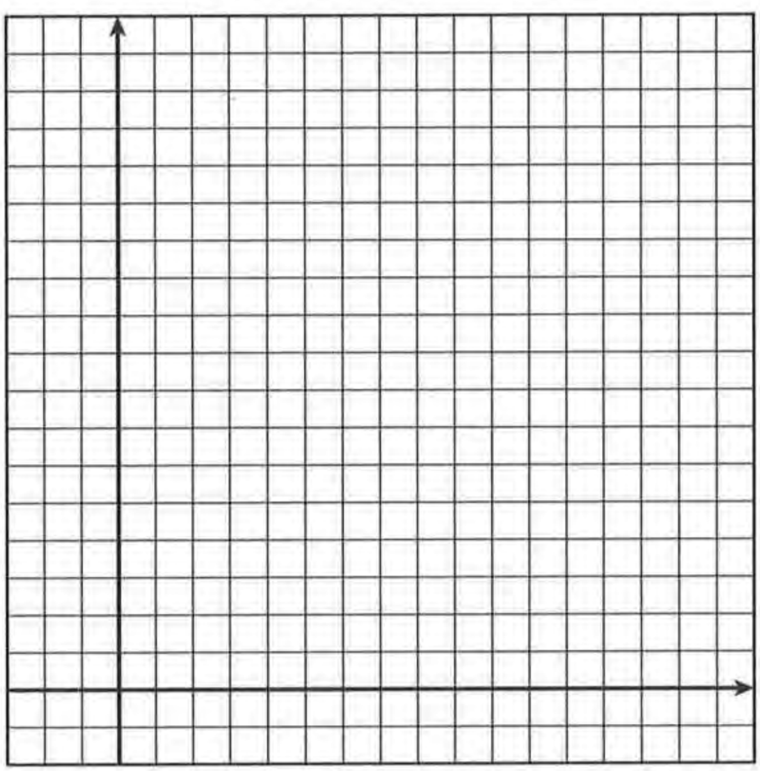
Student's Name **Number of Steps**



xy-coordinate Grid 1
Lesson 2: Problems 19, 24; Unit I Quiz: Problem 12

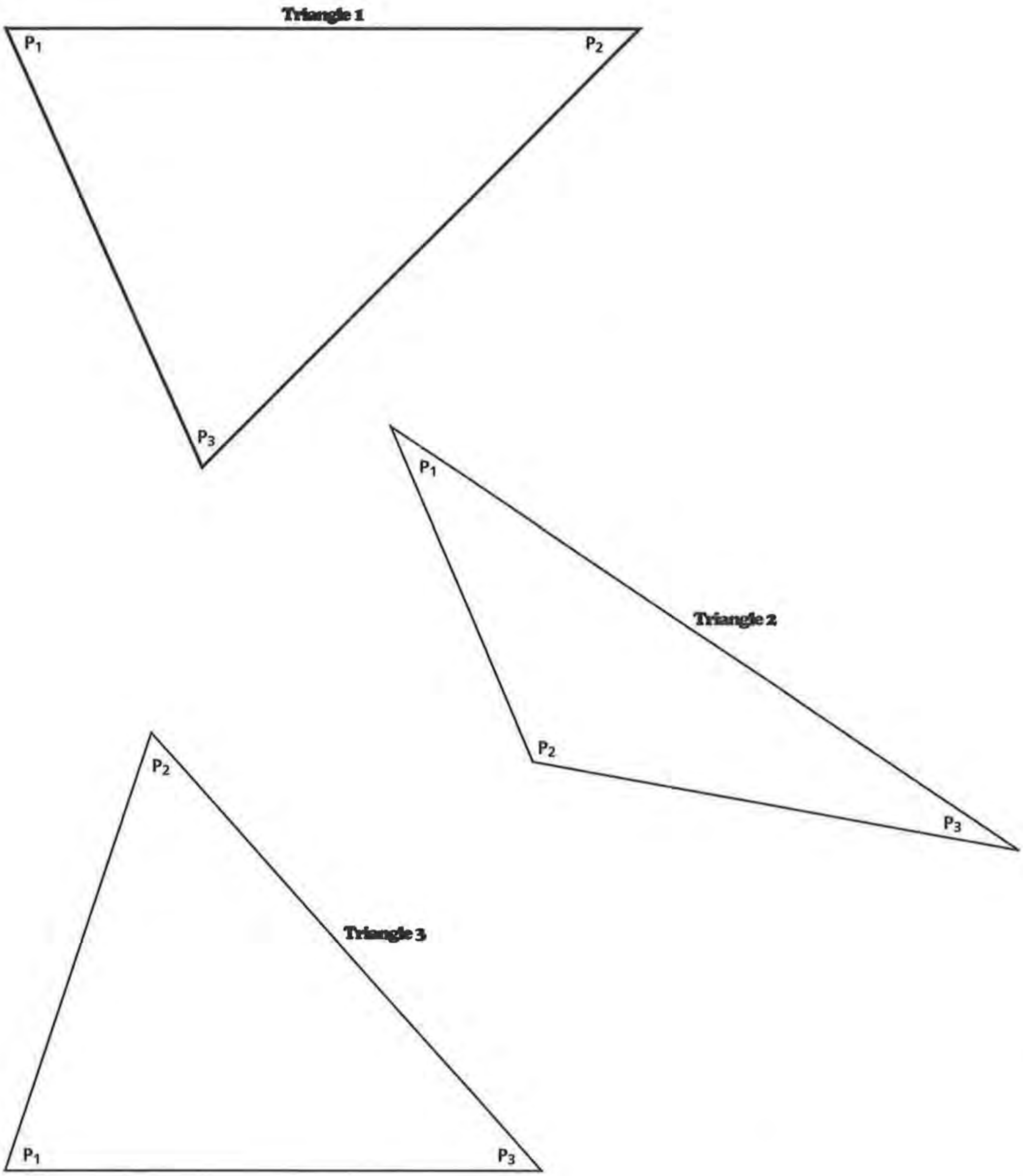
NAME ______________________________





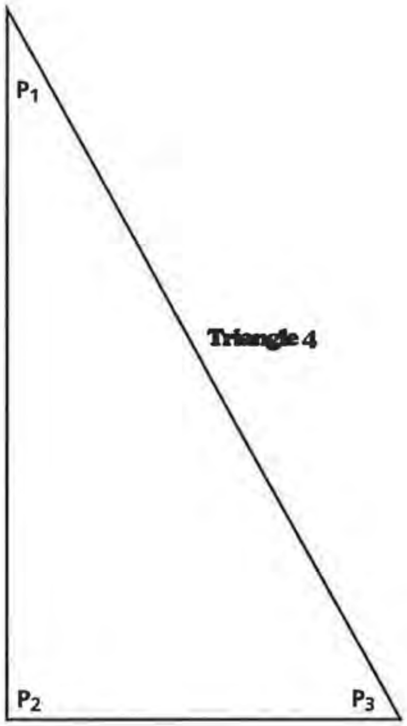
Triangle Options
Lesson 5

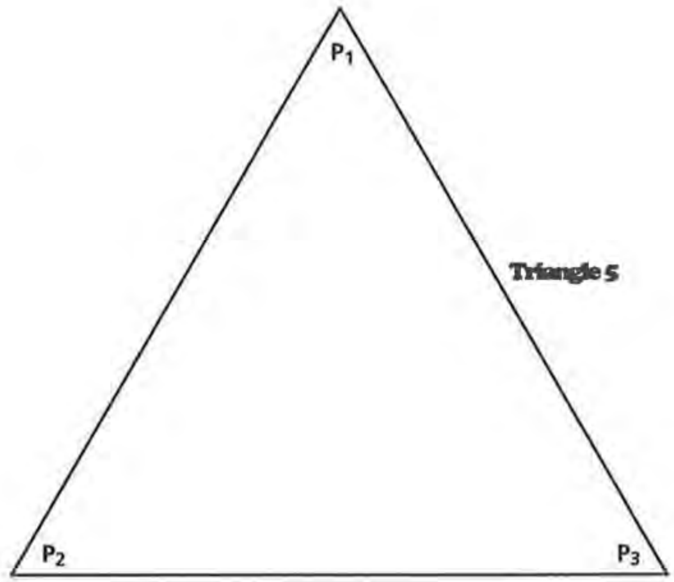
NAME ______________________________



Triangle Options
Lesson 5

NAME ____________________

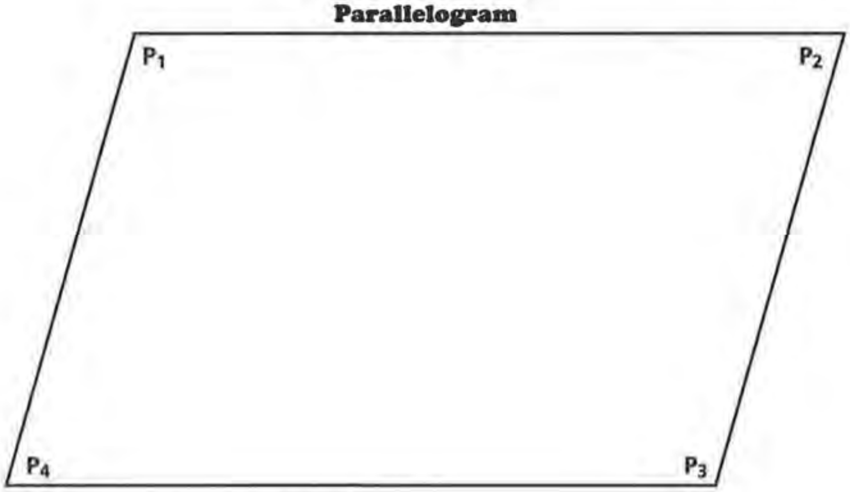




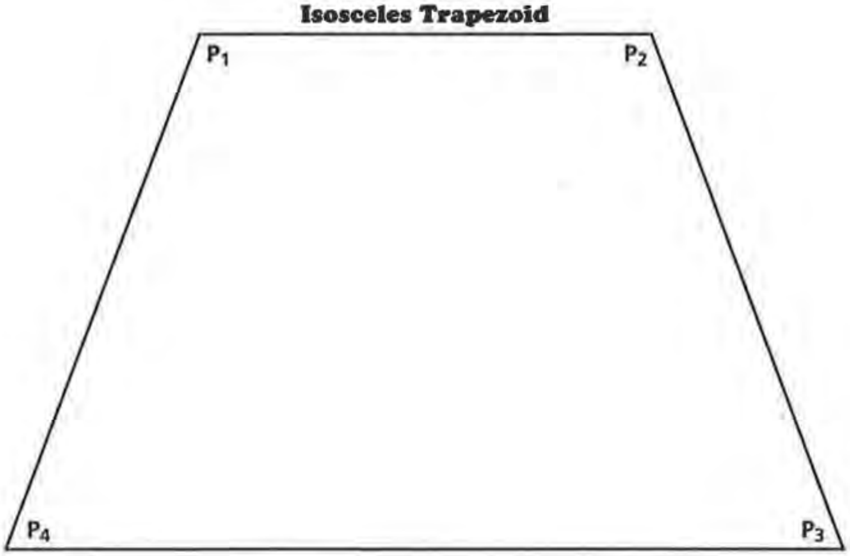
Quadrilateral Options
Lesson 6

NAME ____

Quadrilateral 1
Parallelogram

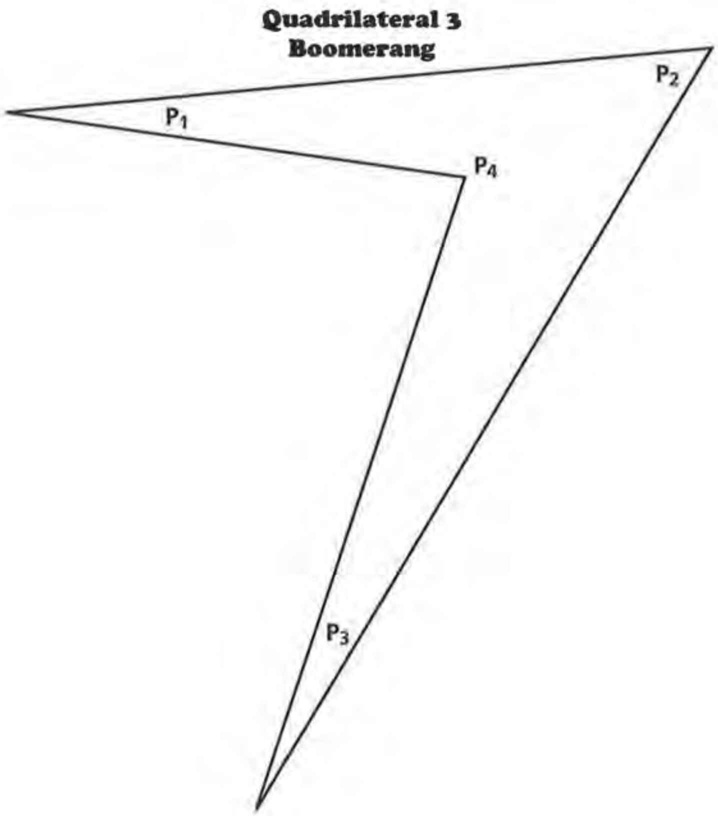


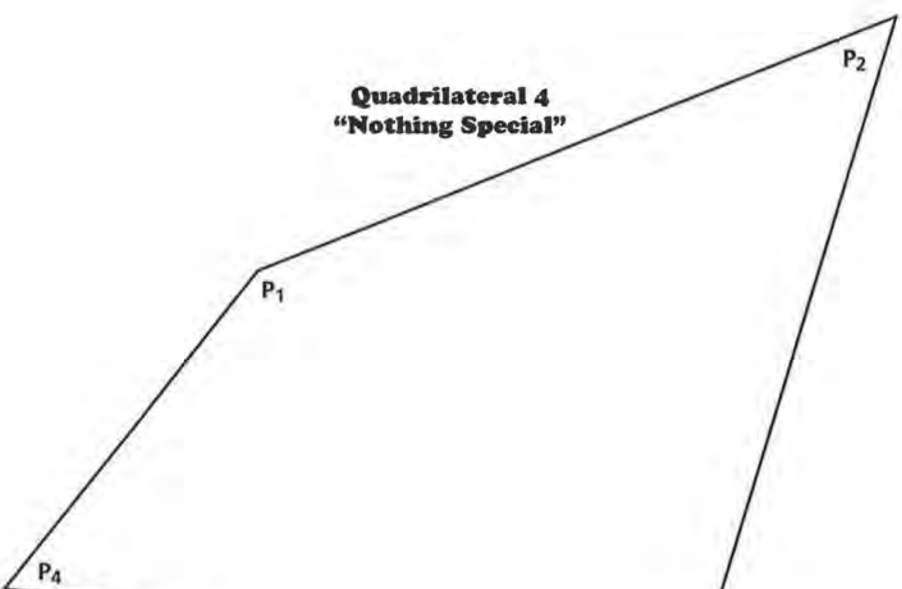
Quadrilateral 2
Isosceles Trapezoid



Quadrilateral Options
Lesson 6

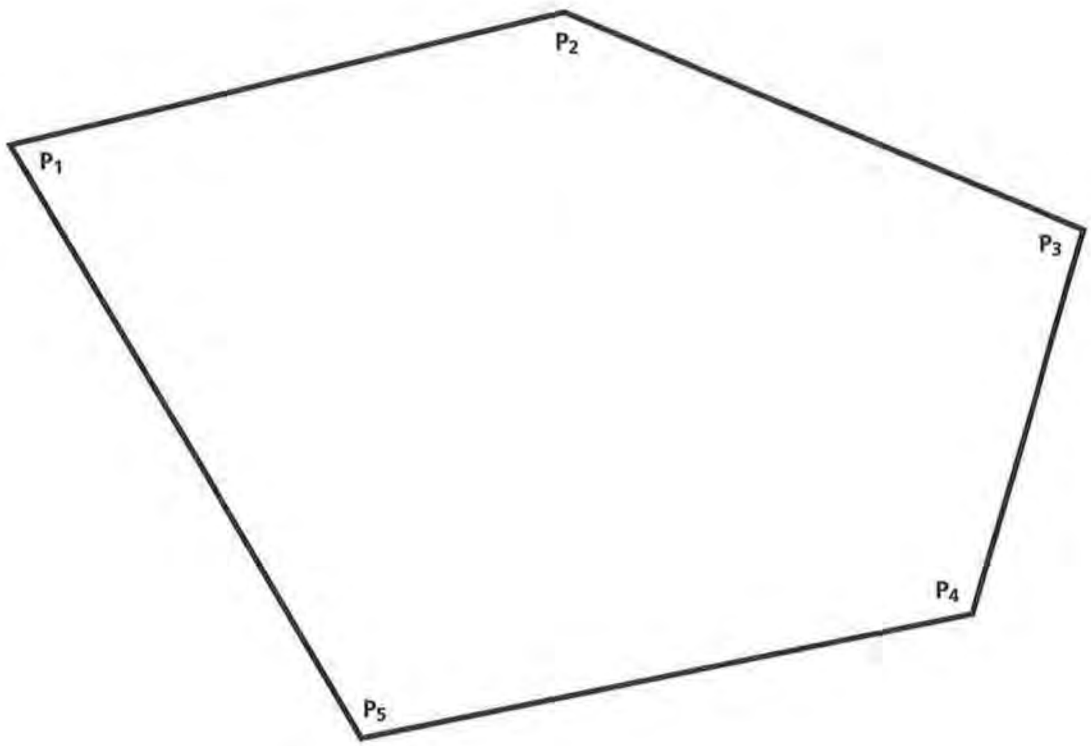
NAME ______________________________





Pentagon Model
Lesson 7

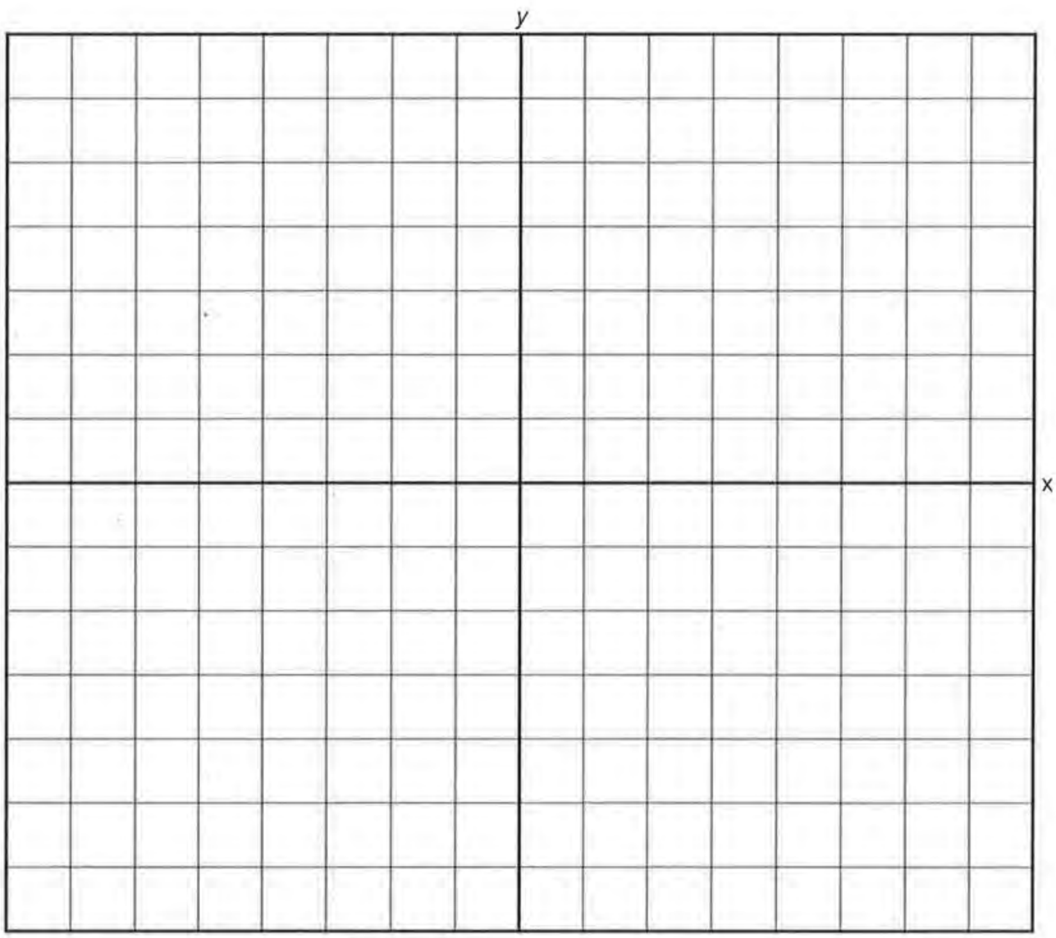
NAME ______________________________



ACTIVITY SHEET 7

xy-coordinate Grid 2
Lessons 5–8

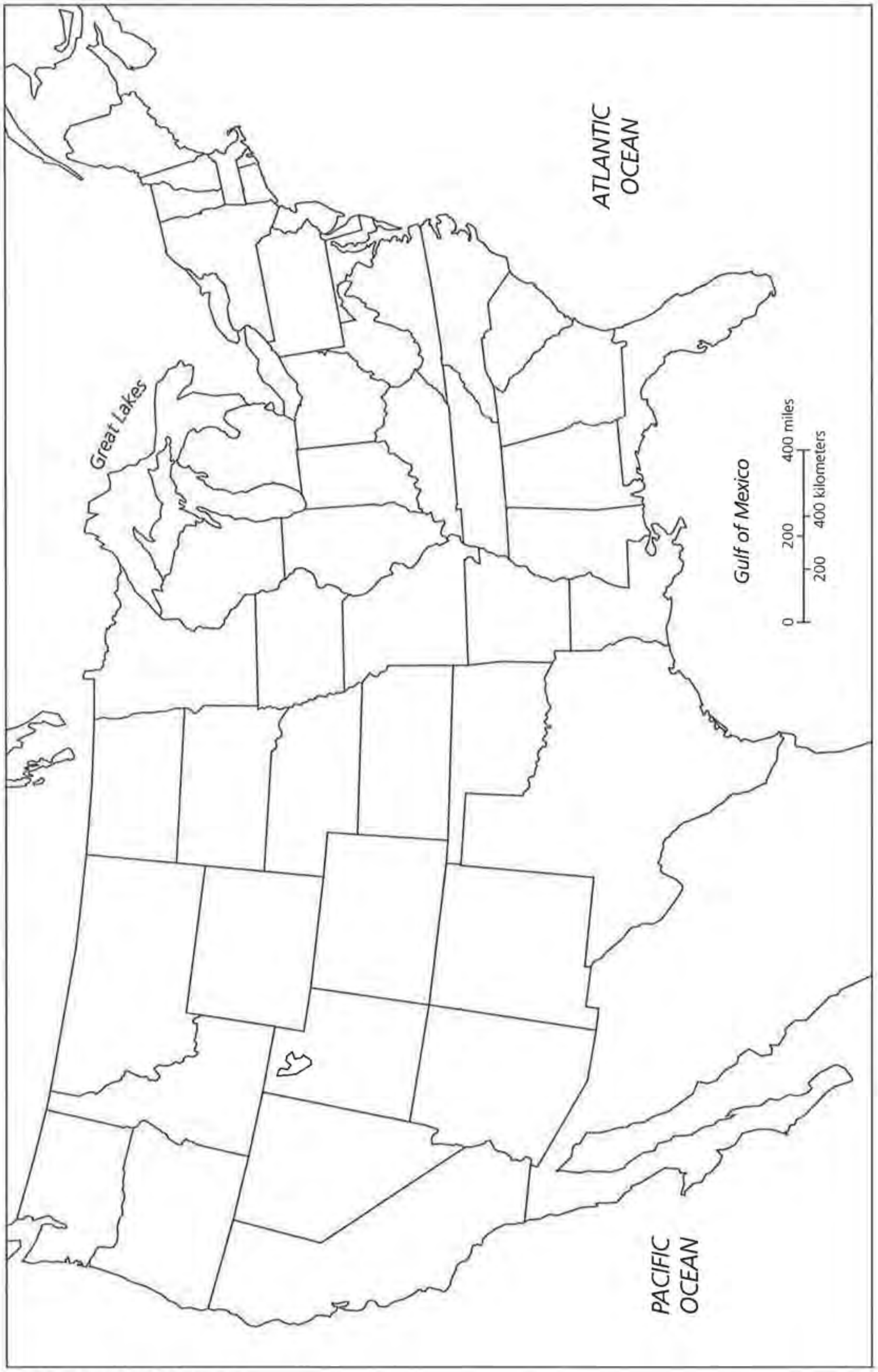
NAME ______________________________



ACTIVITY SHEET 8

U.S. Map: Lesson 10

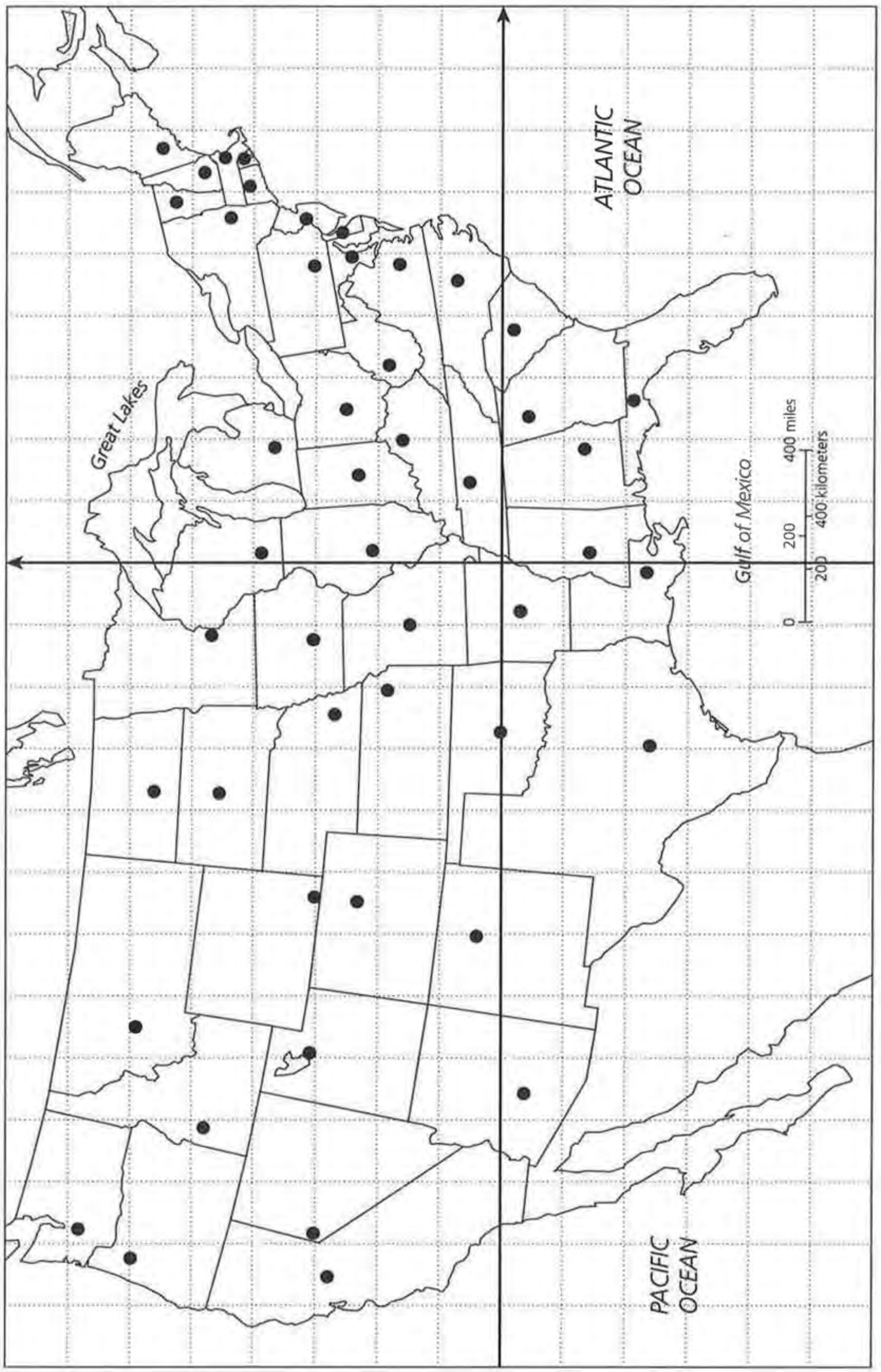
NAME ______________________________



ACTIVITY SHEET 9

U.S. Map with Coordinates
Lesson 10 and Unit IV Quiz

NAME ____________________



Data for U.S. Center Project (A)
Lesson 10

NAME ______________________________

Data for Determining the U.S. Center of Population "The Big Picture"

State (abbreviation)	Population P_i (in thousands)	*x*-coordinate	*y*-coordinate	$P_i x_i$	$P_i y_i$
AK	550				
AL	4,041				
AR	2,351				
AZ	3,665				
CA	29,760				
CO	3,294				
CT	3,287				
DE	666				
FL	12,938				
GA	6,478				
HI	1,108				
IA	2,777				
ID	1,007				
IL	11,431				
IN	5,544				
KS	2,478				
KY	3,685				
LA	4,220				
MA	6,016				
MD	4,781				
ME	1,228				
MI	9,295				
MN	4,375				
MO	5,117				
MS	2,573				

Data for U.S. Center Project (A)
Lesson 10

NAME ______________________________

Data for Determining the U.S. Center of Population
"The Big Picture"

State (abbreviation)	Population P_i (in thousands)	*x*-coordinate	*y*-coordinate	$P_i x_i$	$P_i y_i$
MT	799				
NC	6,629				
ND	639				
NE	1,578				
NH	1,109				
NJ	7,730				
NM	1,515				
NV	1,202				
NY	17,990				
OH	10,847				
OK	3,146				
OR	2,842				
PA	11,882				
RI	1,003				
SC	3,487				
SD	696				
TN	4,877				
TX	16,987				
UT	1,723				
VA	6,187				
VT	563				
WA	4,867				
WI	4,992				
WV	1,793				
WY	454				

Data for U.S. Center Project (B)
Lesson 10

NAME ______________________________

Data for Determining the U.S. Center of Population "The Big Picture"

State (abbreviation)	Population P_i (in thousands)	x-coordinate	y-coordinate	$P_i x_i$	$P_i y_i$
AL	4,041	2	–1.5		
AR	2,351	–0.5	–0.5		
AZ	3,665	–8.5	–0.5		
CA	29,760	–11.5	2.5		
CO	3,294	–5.5	2.5		
CT	3,287	6	4		
DE	666	5.5	2.5		
FL	12,938	2.5	–2		
GA	6,478	2.5	–0.5		
IA	2,777	–1.5	3		
ID	1,007	–9	5		
IL	11,431	0.5	2		
IN	5,544	1.5	2.5		
KS	2,478	–2.5	2		
KY	3,685	2	1.75		
LA	4,220	0	–2.5		
MA	6,016	6.5	4.5		
MD	4,781	5.0	2.5		
ME	1,228	7	5.5		
MI	9,295	2	3.5		
MN	4,375	–1.5	5		
MO	5,117	–1	1.5		
MS	2,573	0.5	–1.5		
MT	799	–7.5	6		

Data for U.S. Center Project (B)
Lesson 10

NAME ______________________________

Data for Determining the U.S. Center of Population "The Big Picture"

State (abbreviation)	Population P_i (in thousands)	x-coordinate	y-coordinate	$P_i x_i$	$P_i y_i$
NC	6,629	4.5	1		
ND	639	–3.5	5.5		
NE	1,578	–2.5	2.5		
NH	1,109	6.5	5		
NJ	7,730	5.5	3.25		
NM	1,515	–6	0.5		
NV	1,202	–10.5	3		
NY	17,990	5.5	4.5		
OH	10,847	2.5	2.5		
OK	3,146	–3	0		
OR	2,842	–11.3	6		
PA	11,882	5	3		
RI	1,003	6.5	4.25		
SC	3,487	4	0		
SD	696	–3.5	4.5		
TN	4,877	1.5	0.5		
TX	16,987	–3.0	–2.5		
UT	1,723	–7.75	3		
VA	6,187	5	1.5		
VT	563	6	5.5		
WA	4,867	–10.5	7		
WI	4,992	0.5	4		
WV	1,793	3.5	2		
WY	454	–5.5	3		

Summary of U.S. Center of Population

NAME ____________________

Population Data Sets of the United States
Compiled from the Statistical Abstract of the United States (1991)

State (abbreviation)	Population P_i 1960 (in thousands)	Population P_i 1970 (in thousands)	Population P_i 1980 (in thousands)	Population P_i 1990 (in thousands)
AK	226	303	402	550
AL	3,267	3,444	3,894	4,041
AR	1,786	1,923	2,286	2,351
AZ	1,302	1,775	2,718	3,665
CA	15,717	19,971	23,668	29,760
CO	1,754	2,210	2,890	3,294
CT	2,535	3,032	3,108	3,287
DE	446	548	594	666
FL	4,952	6,791	9,746	12,938
GA	3,943	4,588	5,463	6,478
HI	633	770	965	1,108
IA	2,758	2,825	2,914	2,777
ID	667	713	944	1,007
IL	10,081	11,110	11,494	11,431
IN	4,662	5,195	5,490	5,544
KS	2,179	2,249	2,364	2,478
KY	3,038	3,221	3,661	3,685
LA	3,257	3,645	4,206	4,220
MA	5,149	5,689	5,737	6,016
MD	3,101	3,924	4,217	4,781
ME	969	994	1,125	1,228
MI	7,823	8,882	9,262	9,295
MN	3,414	3,806	4,076	4,375
MO	4,320	4,678	4,917	5,117
MS	2,178	2,217	2,521	2,573

Summary of U.S. Center of Population

NAME ______________________________

Population Data Sets of the United States
Compiled from the Statistical Abstract of the United States (1991)

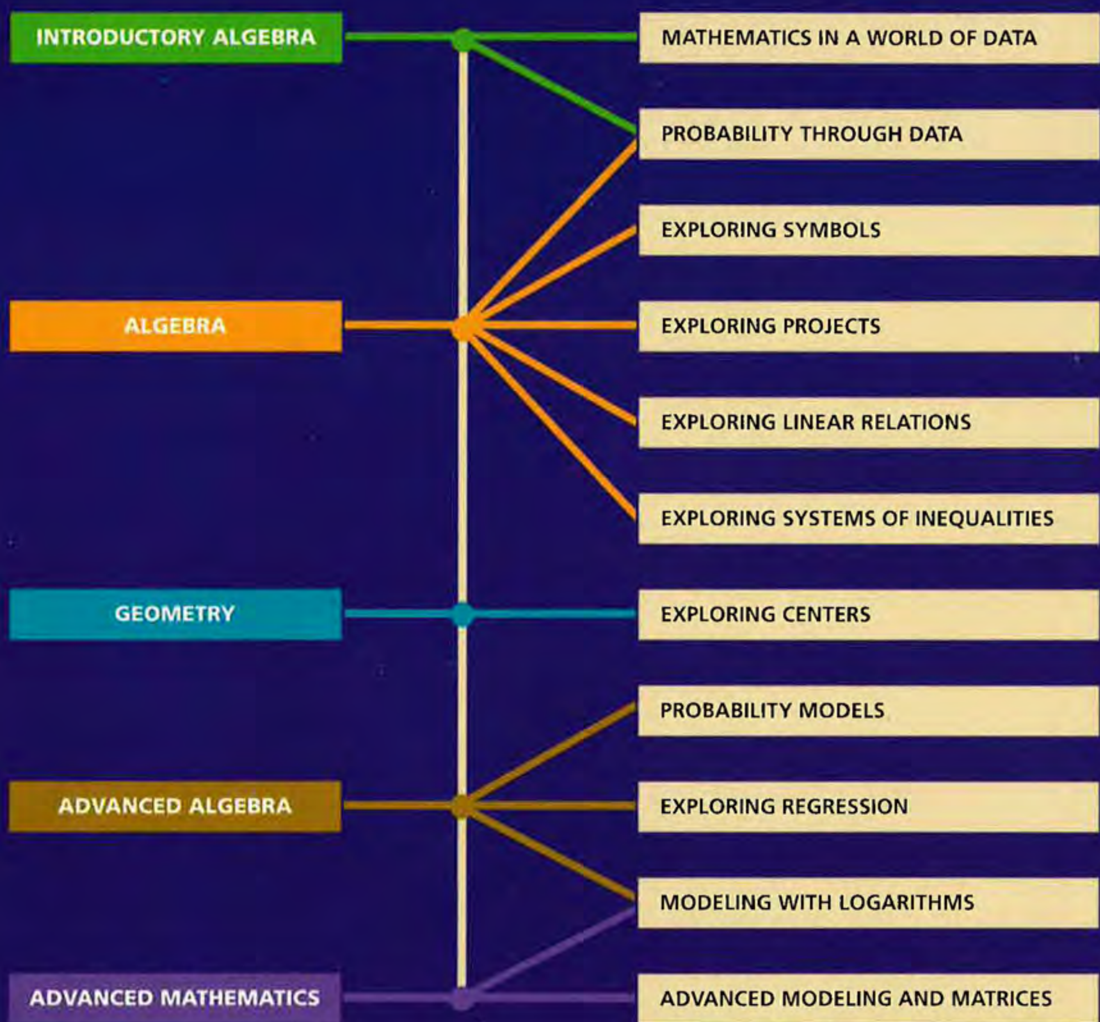
State (abbreviation)	Population P_i 1960 (in thousands)	Population P_i 1970 (in thousands)	Population P_i 1980 (in thousands)	Population P_i 1990 (in thousands)
MT	647	694	787	799
NC	4,556	5,084	5,882	6,629
ND	632	618	653	639
NE	1,411	1,485	1,570	1,578
NH	607	738	921	1,109
NJ	6,067	7,171	7,365	7,730
NM	951	1,017	1,303	1,515
NV	285	489	800	1,202
NY	16,782	18,241	17,558	17,990
OH	9,706	10,657	10,761	10,847
OK	2,328	2,559	3,025	3,146
OR	1,769	2,092	2,633	2,842
PA	11,319	11,801	11,864	11,882
RI	859	950	947	1,003
SC	2,383	2,591	3,122	3,487
SD	681	666	691	696
TN	3,567	3,926	4,591	4,877
TX	9,580	11,199	14,229	16,987
UT	891	1,059	1,461	1,723
VA	3,967	4,651	5,347	6,187
VT	390	445	511	563
WA	2,853	3,413	4,132	4,867
WI	3,952	4,418	4,706	4,992
WV	1,860	1,744	1,950	1,793
WY	330	332	470	454

ACTIVITY SHEET 13

3 × 3 Grid for Stores and 5 × 5 Grid for Stores
Lessons 12–14

NAME ____________________

Data-Driven Mathematics is a series of modules written by teachers and statisticians that focuses on the use of real data and statistics to motivate traditional mathematics topics. This chart suggests which modules could be used to supplement specific middle-school and high-school mathematics courses.



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